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Abstract

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PHYSICS

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GRAVITATIONAL EQUATIONS AND COORDINATE CONDITIONS IN CONFORMAL SPACE

(Presented by Academician V. A. Fock on 11 XII 1965)

The metric of a real three-dimensional space is determined with the aid of the three-dimensional tensor (see (1))

$$a_{ik} = -g^{ik} + g_{0i}g_{0k}/g_{00}, \quad (1)$$

where g_{00}, g_{0i}, g_{ik} are components of the fundamental tensor $g_{\mu\nu}$ of four-dimensional space-time. Latin indices take the values 1, 2, 3, and Greek ones 0, 1, 2, 3. The components of the tensor a_{ik} are assumed to be functions not only of the spatial coordinates x_1, x_2, x_3 , but also of the time coordinate $x_0 = t$, so that the metric of the three-dimensional space is regarded as changing in time.

Let us represent Einstein's gravitational equations in terms of the three-dimensional tensor analysis applied to the conformal space with fundamental tensor

$$h_{ik} = P \cdot a_{ik}, \quad (2)$$

where P is some function of the variables x_1, x_2, x_3 and t .

We shall start from the following contravariant form of the gravitational equations:

$$R^{\mu\nu} = -\kappa (T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T), \quad (3)$$

where $R^{\mu\nu}$ is the four-dimensional curvature tensor, κ is Einstein's gravitational constant, $T^{\mu\nu}$ is the mass tensor, and T is its invariant.

For the tensor $R^{\mu\nu}$ we shall use the expression derived in the book ⁽¹⁾, namely

$$R^{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\frac{\partial^2 g^{\mu\nu}}{\partial x_\alpha\partial x_\beta} - \Gamma^{\mu\nu} + g^{\alpha\sigma}g^{\beta\tau}\Gamma_{\alpha\beta}^\mu\Gamma_{\sigma\tau}^\nu. \quad (4)$$

Here

$$\Gamma^{\mu\nu} = \frac{1}{2}\left(g^{\mu\alpha}\frac{\partial\Gamma^\nu}{\partial x_\alpha} + g^{\nu\alpha}\frac{\partial\Gamma^\mu}{\partial x_\alpha} - \frac{\partial g^{\mu\nu}}{\partial x_\alpha}\Gamma^\alpha\right), \quad (5)$$

$$\Gamma^\nu = g^{\alpha\beta}\Gamma_{\alpha\beta}^\nu, \quad (6)$$

$$\Gamma_{\alpha\beta}^\nu = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\alpha\mu}}{\partial x_\beta} + \frac{\partial g_{\beta\mu}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\mu}\right). \quad (7)$$

Summation is assumed over identical Greek indices from 0 to 3.

Taking into account that g^{00} is a three-dimensional scalar, g^{0i} is a three-dimensional vector, and g^{ik} is a three-dimensional tensor, we have:

$$\nabla_l g^{00} = \frac{\partial g^{00}}{\partial x_l}, \quad (8)$$

$$\nabla_l g^{0i} = \frac{\partial g^{i0}}{\partial x_l} + g^{0m}(\Gamma_{lm}^i)_h, \quad (9)$$

$$\Delta_l g^{ik} = \frac{\partial g^{ik}}{\partial x_l} + g^{im}(\Gamma_{lm}^k)_h + g^{km}(\Gamma_{lm}^i)_h, \quad (10)$$

where

$$(\Gamma_{lm}^i)_h = \frac{1}{2}h^{ik}\left(\frac{\partial h_{kl}}{\partial x_m} + \frac{\partial h_{km}}{\partial x_l} - \frac{\partial h_{lm}}{\partial x_k}\right) \quad (11)$$

are the three-dimensional Christoffel symbols, and by ∇_l are denoted the tensor derivatives in the conformal three-dimensional space (h) with fundamental tensor h_{ik} . Summation over identical Latin indices is assumed from 1 to 3. The contravariant three-dimensional tensor h^{ik} is defined by the equations

$$h_{il}h^{lk} = \delta_{ik}, \quad (12)$$

where $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$, so that

$$g_{00}g^{00} = 1 - \frac{p}{g_{00}} h^{ik} g_{0i} g_{0k}, \quad g^{0i} = \frac{P}{g_{00}} h^{ik} g_{0k}, \quad (13)$$

$$g^{ik} = -Ph^{ik}. \quad (14)$$

Taking (14) into account, we can rewrite equality (10) in the form

$$\nabla_l g^{ik} = -h^{ik} \nabla_l P. \quad (15)$$

Here

$$\nabla_l P = \partial P / \partial x_l, \quad (16)$$

since the function P is a three-dimensional scalar.

Moreover, since $\partial g^{00} / \partial t$ is a three-dimensional scalar, $\partial g^{0i} / \partial t$ is a three-dimensional vector, and $\partial h^{ik} / \partial t$ is a three-dimensional tensor:

$$\frac{\partial}{\partial t} \nabla_l g^{00} - \nabla_l \frac{\partial g^{00}}{\partial t} = 0, \quad (17)$$

$$\frac{\partial}{\partial t} \nabla_l g^{0i} - \nabla_l \frac{\partial g^{0i}}{\partial t} = g^{0m} \frac{\partial (\Gamma_{lm}^i)_h}{\partial t}, \quad (18)$$

$$\nabla_l \frac{\partial h^{ik}}{\partial t} = -h^{im} \frac{\partial (\Gamma_{lm}^k)_h}{\partial t} - h^{km} \frac{\partial (\Gamma_{lm}^i)_h}{\partial t}, \quad (19)$$

and, although the Christoffel symbols $(\Gamma_{kl}^i)_h$ do not have tensor properties, their time derivatives $\partial (\Gamma_{kl}^i)_h / \partial t$ form a three-dimensional tensor.

We denote by Δ the Laplace operator in the conformal space (h) , i.e., we set

$$\Delta = h^{lm} \nabla_l \nabla_m; \quad (20)$$

then

$$\Delta g^{00} = h^{lm} \frac{\partial^2 g^{00}}{\partial x_l \partial x_m} - (\Gamma^l)_h \frac{\partial g^{00}}{\partial x_l}, \quad (21)$$

$$\begin{aligned} \Delta g^{0i} &= h^{lm} \frac{\partial^2 g^{0i}}{\partial x_l \partial x_m} + g^{0k} h^{lm} \frac{\partial (\Gamma_{kl}^i)_h}{\partial x_m} - (\Gamma^l)_h \frac{\partial g^{0i}}{\partial x_l} \\ &\quad + 2h^{lm} (\Gamma_{kl}^i)_h \frac{\partial g^{0k}}{\partial x_m} + g^{0k} h^{lm} (\Gamma_{ln}^i - \Gamma_{km}^n - \Gamma_{kn}^i \Gamma_{lm}^n)_h, \end{aligned} \quad (22)$$

$$\Delta g^{ik} = -h^{ik} \Delta P, \quad (23)$$

where

$$(\Gamma^l)_h = g^{ik} (\Gamma_{ik}^l)_h, \quad (24)$$

$$\Delta P = h^{lm} \frac{\partial^2 P}{\partial x_l \partial x_m} - (\Gamma^l)_h \frac{\partial P}{\partial x_l}. \quad (25)$$

We shall now express the components of the four-dimensional curvature tensor $R^{\mu\nu}$ in terms of tensorial derivatives in the conformal three-dimensional space (h). In doing so we take*

$$\Pi_{kl}^i = \Gamma_{kl}^i - (\Gamma_{kl}^i)_h, \quad (26)$$

$$\Pi_{0\beta}^\alpha = \Gamma_{0\beta}^\alpha, \quad \Pi_{ik}^0 = \Gamma_{ik}^0. \quad (27)$$

Here Π_{00}^0 is a three-dimensional scalar; Π_{0i}^0 and Π_{00}^i are three-dimensional vectors; Π_{ik}^0 , Π_{0k}^i , and Π_{kl}^i are three-dimensional tensors. In addition, let

$$\Pi^\alpha = g^{\mu\nu} \Pi_{\mu\nu}^\alpha, \quad (28)$$

so that

$$\Gamma^0 = \Pi^0, \quad (29)$$

$$\Gamma^i = \Pi^i - P(\Gamma^i)_h, \quad (30)$$

where Π^0 is a three-dimensional scalar and Π^i is a three-dimensional vector.

According to (4), for the temporal component of the four-dimensional curvature tensor we have

$$R^{00} = \frac{1}{2} P \Delta g^{00} - g^{0l} \nabla_l \frac{\partial g^{00}}{\partial t} - \frac{1}{2} g^{00} \frac{\partial^2 g^{00}}{\partial t^2} - \Pi^{00} + g^{\alpha\sigma} g^{\beta\tau} \Pi_{\alpha\beta}^0 \Pi_{\sigma\tau}^0. \quad (31)$$

The mixed components of the four-dimensional curvature tensor take the form

$$R^{0i} = \frac{1}{2} P \Delta g^{0i} - g^{0l} \nabla_l \frac{\partial g^{0i}}{\partial t} - \frac{1}{2} g^{00} \frac{\partial^2 g^{0i}}{\partial t^2} - \frac{1}{2} P g^{0l} h^{ik} (R_{kl})_h +$$

$$+\frac{1}{2}Pg^{00}h^{lm}\frac{\partial(\Gamma_{lm}^i)_h}{\partial t}-\Pi^{0i}+g^{\alpha\sigma}g^{\beta\tau}\Pi_{\alpha\beta}^0\Pi_{\sigma\tau}^i. \quad (32)$$

For the spatial components of the four-dimensional curvature tensor we obtain

$$\begin{aligned} R^{ik} = & -\frac{1}{2}h^{ik}\left(P\Delta P-2g^{0l}\nabla_l\frac{\partial P}{\partial t}-g^{00}\frac{\partial^2 P}{\partial t^2}\right)+Pg^{0l}\nabla_l\frac{\partial h^{ik}}{\partial t}+\frac{1}{2}Pg^{00}\frac{\partial^2 h^{ik}}{\partial t^2}+ \\ & +P^2(R^{ik})_h+\frac{1}{2}Pg^{0i}h^{lm}\frac{\partial(\Gamma_{lm}^k)_h}{\partial t}+\frac{1}{2}Pg^{0k}h^{lm}\frac{\partial(\Gamma_{lm}^i)_h}{\partial t}-\Pi^{ik}+ \\ & +g^{00}\frac{\partial P}{\partial t}\frac{\partial h^{ik}}{\partial t}+g^{0l}\nabla_l P\frac{\partial h^{ik}}{\partial t}+g^{\alpha\sigma}g^{\beta\tau}\Pi_{\alpha\beta}^i\Pi_{\sigma\tau}^k. \end{aligned} \quad (33)$$

In formulas (31)–(33)

$$\Pi^{\mu\nu} = \frac{1}{2}\left(g^{0\mu}\frac{\partial\Pi^\nu}{\partial t}+g^{0\nu}\frac{\partial\Pi^\mu}{\partial t}+g^{\mu l}\nabla_l\Pi^\nu+g^{\nu l}\nabla_l\Pi^\mu-\frac{\partial g^{\mu\nu}}{\partial t}\Pi^0-\nabla_l g^{\mu\nu}\Pi^l\right) \quad (34)$$

and $(R_{ik})_h$ is the curvature tensor of the conformal three-dimensional space (h) .

Let us consider the expressions for Π^α (28) in the static case, i.e., under the assumption that g_{00} , h_{ik} , and P do not depend on time, and $g_{0i} = 0$. In this case

$$(\Pi^0)_{\text{st}} = 0, \quad (35)$$

$$(\Pi^i)_{\text{st}} = -\frac{1}{2g^{00}}h^{ik}\nabla_k(Pg^{00}). \quad (36)$$

* To avoid misunderstanding, we note that in the book repeatedly cited ⁽¹⁾ and in the present paper different quantities are denoted by $\Pi_{\mu\nu}^\alpha$.

Consequently, putting

$$Pg^{00} = \text{const}, \quad (37)$$

we obtain

$$(\Pi^i)_{\text{st}} = 0. \quad (38)$$

If in (37) the constant is taken equal to $1/c^2$, then we arrive at the conformal space which, in the static case, is considered in detail in the book ⁽¹⁾. For a field that is not static, the equations

$$\Pi^\alpha = 0 \quad (39)$$

could be adopted as coordinate conditions. These conditions, unlike harmonic coordinate conditions, have a tensorial character with respect to arbitrary transformations of the spatial coordinates x_1, x_2, x_3 .

Equations (3), with the left-hand side in the form (31)–(33), give the equations of gravitation in conformal space. If, moreover, one uses the coordinate conditions (39), then 6 of Einstein's 10 gravitational equations may be regarded as equations determining the conformal space, and the remaining 4 as equations determining the gravitational field in this conformal space. At the same time, since the quantities g^{00} , g^{0i} , and h^{ik} , and their first derivatives, enter into all the gravitational equations, the conformal space proves to depend on the quantities g^{00} and g^{0i} , and conversely; i.e., the determination of the conformal space and of the field must be carried out jointly.

For a quasistatic field, in the first approximation one may put (see ⁽¹⁾)

$$g^{00} = \frac{1}{c^2} \left(1 + \frac{2U}{c^2} \right), \quad (40)$$

$$g^{0i} = \frac{4U_i}{c^4}, \quad (41)$$

$$h^{ik} = \delta_{ik}. \quad (42)$$

The components g^{ik} are determined by formula (14), where

$$P = \frac{1}{c^2 g^{00}}. \quad (43)$$

In the formulas given, U is the Newtonian potential and U_i is the vector potential of the gravitational field. In the indicated approximate solution the conformal space is flat, and the coordinates are Cartesian. The coordinate conditions (39) in this case coincide with the harmonic ones.

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CITED LITERATURE

1. V. A. Fock, *The Theory of Space, Time, and Gravitation*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.

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