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Abstract

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MATHEMATICAL PHYSICS

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ON THE REPRESENTATION OF STATES IN QUANTUM FIELD THEORY

(Presented by Academician N. N. Bogolyubov on 16 XII 1965)

For the investigation of certain general properties of the S -matrix for arbitrary spin it is necessary to consider the problem of describing a dynamical system consisting of several noninteracting quantized fields characterized by operator-valued generalized functions $\hat{u}_1, \dots, \hat{u}_n$.

We introduce into consideration the countably normed Hilbert space ⁽¹⁾

$$D_S = \bigotimes_{n,l=0}^{\infty} D_S^{(n,l)}(R^4(n,l)) \quad (1)$$

of regular states $|\Phi\rangle$, where $D_S^{(n,l)}(R^4(n,l))$ is the space of states of a bundle of non-interacting n particles and l antiparticles, and, for spins $s_r, j_t = 0, \frac{1}{2}, 1, \dots$, $r = 1, \dots, n$, $t = 1, \dots, l$, the space $D_S^{(n,l)}(R^4(n+l))$ consists of elements of the space $S(R^{4(n+l)})$ of basic functions ⁽²⁾, represented in the form

$$\Phi_{(\sigma)n;(\nu)l}^{(n,l)}(p, q)_{n,l} = \bigotimes_{r=1}^n \mathcal{D}_{\sigma_r \sigma_r}^{s_r}(l^{-1}(p_r)) \bigotimes_{t=1}^l \mathcal{D}_{\nu'_t \nu_t}^{j_t}(l^{-1}(q_t)) \hat{\Phi}_{(\sigma')n;(\nu')l}^{(n,l)}(p, a)_{n,l}, \quad (2)$$

where

$$\hat{\Phi}_{(\sigma)n;(\nu)l}^{(n,l)}(p, a)_{n,l} = \left(\bigotimes_{r=1}^n U(l(p_r)) \bigotimes_{t=1}^l U(l(q_t)) \Phi \right)_{(\sigma,\nu)n,l}^{(n,l)};$$

σ_r and ν_t are the spin projections, respectively, of a particle of mass m_r and spin s_r and of an antiparticle of mass μ_t and spin j_t on the third axis; p_r and q_t are their 4-energy-momenta with domain of definition

$$\Omega_{n,l} = \bigotimes_{r=1}^n \Omega_{m_r} \bigotimes_{t=1}^l \Omega_{\mu_t},$$

where $\Omega_m, p_r^2 = m_r^2, p_{0r} > 0, r = 1, \dots, n$, and $\Omega_{\mu_t}: q_t^2 = \mu_t^2, q_{0t} > 0, t = 1, \dots, l$; $D_{\sigma'\sigma}(l(p))$ is a unitary irreducible representation of the $(2s+1)$ -dimensional spin space $^{(3)}$, $l(p) \in SL(2, c)$ and $l(p)\tilde{p} = p, \tilde{p} = (m, \vec{0}), (k)_f = (k_1, \dots, k_f)$. We shall assume that $\Phi^{(n,l)} = 0$ for $n, l > \tilde{N}$, where $\tilde{N}$ is a sufficiently large number. A topology is introduced in D_S in a certain way.

The transformation property of the elements $|\Phi\rangle \in D_S$ with respect to the spinor Poincaré group $\hat{P}_+^\uparrow$ with elements (a, A) , where a denotes 4-translations and A is an arbitrary element of $SL(2, c)$ corresponding to the element $\Lambda \in L_+^\uparrow$, is defined according to

$$U(a, A)|\Phi\rangle = \left\{ \exp \left[ia \left(\sum_r p_r + \sum_t q_t \right) \right] \times \right. \\ \left. \times \bigotimes_{r=1}^n \mathcal{D}_{\sigma'_r \sigma_r}^{s_r}(A) \bigotimes_{t=1}^l \mathcal{D}_{\nu'_t \nu_t}^{j_t}(A) \Phi_{(\sigma')n; (\nu')l}^{(n,l)}(A^{-1}p; A^{-1}q)_{n,l} \right\}. \quad (3)$$

The latter follows from the fact that

$$(U(a, A)\widehat{\Phi})_{(\sigma)n; (\nu)l}^{n,l}(p; q)_{n,l} = \exp \left[ia \left(\sum_r p_r + \sum_t q_t \right) \right] \times \\ \times \bigotimes_{r=1}^n \mathfrak{D}_{\sigma'_r \sigma_r}^{s_r}(R(A)p_r) \bigotimes_{t=1}^l \mathfrak{D}_{\nu'_t \nu_t}^{j_t}(R(A, q_t)) \widehat{\Phi}_{(\sigma')n; (\nu')l}^{(n,l)}(A^{-1}p; A^{-1}q)_{n,l}, \quad (4)$$

where $R(A, p) = l^{-1}(A^{-1}p)Al(p) \in SU(2, c)$.

In D_S the scalar product is introduced

$$\langle \Phi_1 | \Phi_2 \rangle = \sum_{n,l}^\infty \langle \Phi_1^{(n,l)} | \Phi_2^{(n,l)} \rangle < \infty, \quad (5)$$

where

$$\langle \Phi_1^{(n,l)} | \Phi_2^{(n,l)} \rangle = \langle (U(a, A)\Phi)^{(n,l)} | (U(a, A)\Phi_2)^{(n,l)} \rangle. \quad (6)$$

Let H_{L_2} be the Hilbert space obtained by completing D_S with respect to the nondegenerate scalar product (5). Then the family of spaces $D_S \subset H_{L_2} \subset D'_S$,

where D'_S is the space of generalized states, forms a rigged physical Hilbert space $(^1, ^4)$.

The space $D_{S(R^4)}^{(1,0)'}(m, s)$ of one-particle generalized states

$$\eta^{(1)}(\Phi^{(1)}) = \eta_{\sigma}^{(1)}(\Phi_{\sigma}^{(1)}) = \int \eta_{\sigma}^{(1)}(p) \Phi_{\sigma}^{(1)}(p) \frac{d^3 p}{p}, \quad (7)$$

where $\Phi_{\sigma}^{(1)}(p) \in D_{S(R^4)}^{(1,0)}(m, s)$, is the space of irreducible representations of the spinor Poincaré group $\tilde{P}_+ \uparrow$. Generalized states are defined as common generalized eigenfunctions of the operators $P'_{\mu}, \vec{S}'^2, S'_3$ and form a canonical basis of the spinor space $(^5)$. L. Schwartz' s kernel theorem $(^2)$ makes it possible to define many-particle generalized states in the form

$$\rho_{(\sigma)_n}^{(n)}(\Phi_{(\sigma)_n}^{(n)}) = \eta_{(\sigma)_n}^{(n)}(\Phi_{\sigma_1}^{(1)}, \dots, \Phi_{\sigma_n}^{(1)}) = \eta_{\sigma_1}^{(1)}(\Phi_{\sigma_1}^{(1)}) \dots \eta_{\sigma_n}^{(1)}(\Phi_{\sigma_n}^{(1)}), \quad (8)$$

where $\Phi_{(\sigma)_n}^{(n)}(p)_n = \Phi_{\sigma_1}^{(1)}(p_1) \dots \Phi_{\sigma_n}^{(1)}(p_n)$. Therefore the vectors of an arbitrary generalized state $|\rho(\Phi)\rangle \in D'_S$ can be represented in the form

$$|\rho(\Phi)\rangle = \sum'_{n,l \geq 0} \int_{\Omega_{nl}} \dots \int \rho_{(\sigma)_n;(\nu)_l}^{(n,l)}(p, q)_{n,l} \Phi_{(\sigma)_n;(\nu)_l}^{(n,l)}(p, q)_{n,l} \prod_{r=1}^n \frac{d^3 p_r}{p_{0r}} \prod_{t=1}^l \frac{d^3 q_t}{q_{0t}} \quad (9)$$

and possess the transformation property

$$U(a, A) |\rho(\Phi)\rangle = \{\rho_{(\sigma)_n;(\nu)_l}^{(n,l)}((U(a, A)\Phi)_{(\sigma)_n;(\nu)_l}^{(n,l)}(p, q)_{n,l})\}, \quad (10)$$

where $(U(a, A)\Phi)^{(n,l)}$ is defined according to (3).

In the space D'_S one can define weakly dense sets of vectors of generalized correlations of a special type. We shall present a scheme for constructing such a set.

To each function $f(x) \in S(R^4)$ we assign the Hermitian operator

$$A(f) = \int d^4 x f(x) A(x) \text{ in } D_S, \quad \langle \Psi | A(f) | \Phi \rangle, \quad \text{where } \Psi, \Phi \in D_S. \quad (11)$$

and we shall consider the quantity $A(f)$ as a linear continuous functional in $S(R^4)$ and call it an **operator-valued generalized function**.

Let us now consider the field operator $\chi(f)$ for the creation of a particle of mass $m \neq 0$ and spin s , defined as follows:

$$\chi(\tilde{f}) = \chi_\sigma(\tilde{f}^\sigma) = \int \frac{d^3 p}{p_0} \tilde{f}^\sigma(p) \chi_\sigma(p), \quad \text{where} \quad \tilde{f}^\sigma(p) = \int e^{-ipx} f^\sigma(x) d^4 x; \quad (12)$$

$$\begin{aligned} \chi(f)|\Phi\rangle &= \left\{ \frac{1}{\sqrt{n}} \sum_{r=1}^n \sum_{\sigma_r=-s_r, \dots, s_r} \tilde{f}^{\sigma_r}(-p_r) \times \right. \\ &\times \Phi_{\sigma_1, \dots, \sigma_{r-1}, \sigma_{r+1}, \dots, \sigma_n; (\nu)_l}^{(n-1, l)}(p_1, \dots, p_{r-1}, p_{r+1}, \dots, p_n; (q)_l) \Big\}; \end{aligned} \quad (13)$$

$$U(a, A) \chi_\sigma(\tilde{f}^\sigma) U^{-1}(a, A) = \chi_\sigma(e^{-iaAp} \mathcal{D}_{\sigma', \sigma}^s(R^{-1}(A, Ap)) \tilde{f}^{\sigma'}(Ap)) \quad (14)$$

for any $f^\sigma(x) \in S(R^4)$ and $|\Phi\rangle \in D_S$.

We construct the corresponding covariant operator by setting

$$a(\tilde{f}) = a_\alpha(\tilde{f}^\alpha) = \chi_\sigma(\mathcal{D}_{\alpha\sigma}^s(l(p)) \tilde{f}^\alpha(p)) \quad (15)$$

with the transformation property

$$U(a, A) a_\alpha(\tilde{f}^\alpha) U^{-1}(a, A) = a_\alpha(e^{-iaAp} \mathcal{D}_\beta^{s\alpha}(A^{-1}) \tilde{f}^\beta(Ap)). \quad (16)$$

Now the free field can be characterized by means of the covariant operator-valued generalized function $\varphi(f)$, defined in D_S :

$$\begin{aligned} \varphi(f)|\Phi\rangle &= a(\tilde{f}(-p))|\Phi\rangle + a^*(\tilde{f}(p))|\Phi\rangle = \\ &= \left\{ \frac{1}{\sqrt{n}} \sum_{r=1}^n \sum_{\sigma_r=-s_r, \dots, s_n} \mathcal{D}_{\alpha_r, \sigma_r}^{s_r}(l(p_r)) \tilde{f}^{\alpha_r}(-p_r) \times \right. \\ &\times \Phi_{\sigma_1, \dots, \sigma_{r-1}, \sigma_{r+1}, \dots, \sigma_n; (\nu)_l}^{(n-1, l)}(p_1, \dots, p_{r-1}, p_{r+1}, \dots, p_n; (q)_l) + \\ &\left. + \frac{1}{\sqrt{n+1}} \int \frac{d^3 p}{p_0} \mathcal{D}_{\sigma\alpha}^s(l^{-1}(p)) \tilde{f}^\alpha(p) \Phi_{\sigma, (\sigma)_n; (\nu)_l}^{(n+1, l)}(p, (p)_n; (q)_l) \right\}. \end{aligned} \quad (17)$$

It is clear that

$$\varphi(f) D_S \subset D_S,$$

$$U(a, A)\varphi_\alpha(f^\alpha)U^{-1}(a, A) = \varphi_\alpha\left(\mathcal{D}_\beta^{s\alpha}(A^{-1})f^\beta(A^{-1}(x-a))\right) \quad (18)$$

for $f^\alpha(x) \in S(R^4)$.

Analogously, introducing the field operator in D_S according to

$$\chi(f) = \chi^{\dot{\beta}}(f_{\dot{\beta}}) = b^{\dot{\beta}}(\tilde{f}_{\dot{\beta}}) + (-1)^{2s} b^{\dot{\beta}*}(\tilde{f}_{\dot{\beta}}), \quad (19)$$

where

$$b(\tilde{f}) = b^{\dot{\beta}}(\tilde{f}_{\dot{\beta}}) = \chi_\sigma\left(\mathcal{D}_{\dot{\beta}}^{s\sigma}(l(p))\tilde{f}_{\dot{\beta}}(p)\right), \quad (20)$$

$$U(a, A)b^{\dot{\beta}}(\tilde{f}_{\dot{\beta}})U^{-1}(a, A) = b^{\dot{\beta}}\left(e^{-iaAp}\mathcal{D}_{\dot{\beta}}^{s\alpha}(A)\tilde{f}_{\dot{\beta}}(Ap)\right). \quad (21)$$

The field operators φ_α and $\chi^{\dot{\beta}}$ satisfy the Dirac equation

$$\varphi_\alpha(\Phi^\alpha) = \chi^{\dot{\beta}}(f_{\dot{\beta}}) \quad (22)$$

for arbitrary functions $\Phi^\alpha(x)$ and $f_{\dot{\beta}}(x)$ from $S(R^4)$ satisfying the relations

$$\begin{pmatrix} 0 & (-1)^{2s}\mathcal{D}^s(i\sigma \cdot \partial)_{\alpha\dot{\beta}} \\ (-1)^{2s}\mathcal{D}^s(i\tilde{\sigma} \cdot \partial)^{\dot{\beta}\alpha} & 0 \end{pmatrix} \begin{pmatrix} \Phi^\alpha(x) \\ f_{\dot{\beta}}(x) \end{pmatrix} = m^{2s} \begin{pmatrix} \Phi^\alpha(x) \\ f_{\dot{\beta}}(x) \end{pmatrix}, \quad (23)$$

where $\sigma = (\sigma_0, \vec{\sigma})$, $\tilde{\sigma} = (\sigma_0, -\vec{\sigma})$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, and σ_0 is the identity matrix.

Now suppose that there exists a set of nonproper field operators $\varphi_{\alpha_1}^{(1)}, \varphi_{\alpha_2}^{(2)}, \dots$, defined in D_S . Then, using nuclear

by theorem (?), we can construct a set of generalized-state vectors everywhere dense in D_S ,

$$\{c_{n,l}\Omega^{*(n,l)}(f_{n,l})|0\rangle\}, \quad (24)$$

where $c_{n,l}$ are arbitrary complex numbers, $|0\rangle \in D_S$ are vacuum states, and

$$\Omega_{(\alpha)n;(\beta)l}^{*(n,l)}(f_{n,l}^{(\alpha)n;(\beta)l}) = a_{\alpha_1}^{*(1)} \dots a_{\alpha_n}^{*(n)} c_{\beta_1}^{*(1)} \dots c_{\beta_l}^{*(l)}(f_{n,l}^{(\alpha)n;(\beta)l}) \quad (25)$$

with the transformation property

$$\begin{aligned}
 U(a, A) \Omega_{(\alpha)n}^{*(n)}(f_n^{(\alpha)n}) U^{-1}(a, A) = \\
 = \Omega_{(\alpha)n}^{*(n)} \left(\exp \left[ia \sum_r A p_r \right] \bigotimes_{r=1}^n \mathcal{D}_{\beta_r \alpha_r}^{s_r}(A^{-1}) f_n^{(\beta)n}(A p)_n \right) \quad (26)
 \end{aligned}$$

for $\tilde{f}_m^{(\alpha)n}(p)_n \in S(R^{4n})$.

In the following work we shall study relativistically invariant matrix elements of the S -matrix, defined in the form:

$$S(\Phi_{M,n}^{(\alpha)m;(\beta)r} \Psi_{n,l}^{(\alpha)n;(\beta)l}) = \langle 0 | \Omega_{(\alpha)m;(\beta)r}^{(m,r)}(\Phi_{m,r}^{(\alpha)m;(\beta)r}) S \Omega_{(\alpha)n;(\beta)l}^{*(n,l)}(\Psi_{n,l}^{(\alpha)n;(\beta)l}) | 0 \rangle. \quad (27)$$

Here we shall regard the S -matrix as a unitary operator defined in the equipped physical Hilbert space $D \subset H \subset D'$, which maps one generalized state (25) from D' , corresponding to an incident wave, into some other generalized state from D' , corresponding to an outgoing wave.

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Note: Figure translations are in progress. See original paper for figures.

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