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Abstract

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MATHEMATICS

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ON THE DIRECT AND INVERSE PROBLEM OF V. A. MARKOV IN THE COMPLEX DO- MAIN

(Presented by Academician A. N. Kolmogorov on 4 V 1965)

1. Let continuous complex-valued functions $f(z)$, $\{\varphi_j(z)\}_0^n$ be given on a bi-compact Hausdorff space G ⁽¹⁾. We regard the functions $\{\varphi_j(z)\}_0^n$ as linearly independent. Consider the following generalization of the work of V. A. Markov ⁽²⁾. Among all polynomials

$$F(a; z) = \sum_{j=0}^n a_j \varphi_j(z), \quad z \in G,$$

whose coefficients satisfy the linearly independent relations

$$\sum_{j=0}^n a_j \alpha_j^{(t)} = \alpha_t, \quad t = 1, \dots, p, \quad (1)$$

find a polynomial $F(a^*; z)$ for which the condition

$$\min_{a_j} \max_{z \in G} \left| \sum_{j=0}^n a_j \varphi_j(z) - f(z) \right| = \max_{z \in G} \left| \sum_{j=0}^n a_j^* \varphi_j(z) - f(z) \right| = \rho \quad (2)$$

is fulfilled.

Polynomials $F(a; z)$ whose coefficients satisfy the relations (1) will be called **admissible**.

Works ⁽²⁻¹⁶⁾ and others are devoted to the problem of Chebyshev approximation without constraints and with constraints.

In the present paper* the results of the author's work ^(11,12) are generalized to problem (2) with constraints (1). The following notation is used:

$$\operatorname{Re} \varphi_j(z) = \varphi_j'(z), \quad \operatorname{Im} \varphi_j(z) = \varphi_j''(z), \quad \varphi_j(z) = \varphi_j'(z) + i\varphi_j''(z),$$

$$\alpha_j^{(t)} = \alpha_j^{(t)'} + i\alpha_j^{(t)''}, \quad \alpha_t = \alpha_t' + i\alpha_t'', \quad \delta(z_s) = F(a; z_s) - f(z_s),$$

$$\operatorname{sgn} \delta(z_s) = e^{i\theta_s}, \quad \theta_s = \arg \delta(z_s), \quad \varphi_j^*(z) = \varphi_j(z) \operatorname{sgn} \overline{\delta(z)},$$

$$\operatorname{sgn} \overline{\delta(z)} = e^{-i\theta_s}, \quad \varphi_j^*(z) = \varphi_j^{*'}(z) + i\varphi_j^{*''}(z), \quad F^*(a; z) = F(a; z) \operatorname{sgn} \overline{\delta(z)}.$$

Consider the systems of equations:

$$\sum_{s=1}^r K_s^{(t)} \varphi_j^{*'}(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)} d_j'(s) = \alpha_j^{(t)'}, \quad j = 0, \dots, n; \quad (3)$$

$$\sum_{s=1}^r K_s^{(t)} \varphi_j^{*''}(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)} d_j''(s) = \alpha_j^{(t)''}, \quad j = 0, \dots, n; \quad t = 1, \dots, p.$$

$$\sum_{s=1}^r K_s^{(t)*} \varphi_j^*(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)*} d_j(s) = -\alpha_j^{(t)''}, \quad j = 0, \dots, n; \quad (4)$$

* The results of the work were reported at a meeting of the Academic Council of the Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR on 23 I 1963.

$$\sum_{s=1}^r K_s^{(t)*} \varphi_j^{*'}(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)*} d_j''(s) = \alpha_j^{(t)'}, \quad j = 0, \dots, n; \quad t = 1, \dots, p.$$

In the systems (3), (4) the number $r \geq 1$, and the $2n + 2$ vectors

$$\vec{\varphi}_s^* = [\varphi_0^{*'}(z_s), \varphi_0^{*''}(z_s), \dots, \varphi_n^{*'}(z_s), \varphi_n^{*''}(z_s)], \quad s = 1, \dots, r;$$

$$\mathbf{d}_s = [d_0'(s), d_0''(s), \dots, d_n'(s), d_n''(s)], \quad s = r + 1, \dots, 2n + 2,$$

are linearly independent, where $d_\nu'(s), d_\nu''(s)$ are real numbers.

Lemma 1. The vectors $\mathbf{K}^{(t)} = [K_1^{(t)}, \dots, K_{2n+2}^{(t)}]$, $\mathbf{K}^{(t)*} = [K_1^{(t)*}, \dots, K_{2n+2}^{(t)*}]$, $t = 1, \dots, p$, are linearly independent.

From (3), (4) we obtain

$$\sum_{s=1}^r K_s^{(t)} \varphi_j^*(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)} d_j(s) = \alpha_j^{(t)}, \quad j = 0, \dots, n; \quad t = 1, \dots, p; \quad (5)$$

$$\sum_{s=1}^r K_s^{(t)*} \varphi_j^*(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)*} d_j(s) = i\alpha_j^{(t)}, \quad j = 0, \dots, n; \quad t = 1, \dots, p. \quad (6)$$

From (5), (6), analogously to what was done in ^(11, 12), we obtain

$$\sum_{s=1}^r K_s^{(t)} \tilde{F}^{*'}(\tilde{A}; z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)} \tilde{\Phi}'(\tilde{A}; s) \equiv 0, \quad t = 1, \dots, p; \quad (7)$$

$$\sum_{s=1}^r K_s^{(t)*} \tilde{F}^{*'}(\tilde{A}; z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)*} \tilde{\Phi}'(\tilde{A}; s) \equiv 0, \quad t = 1, \dots, p, \quad (8)$$

where $\tilde{F}^{*'}(\tilde{A}; z_s)$, $\tilde{\Phi}'(\tilde{A}; s)$ are real forms with free parameters.

Theorem 1. In order that the vectors $\mathbf{K}_s = [K_s^{(1)}, \dots, K_s^{(p)}, K_s^{(1)*}, \dots, K_s^{(p)*}]$, $s = 1, \dots, 2p$, corresponding to the identities (7), (8), be linearly independent, it is necessary and sufficient that the forms $\tilde{F}^{*'}(\tilde{A}; z_l)$, $\tilde{\Phi}'(\tilde{A}; m)$ ($l, m \neq s_1, \dots, s_{2p}$) be linearly independent.

Let $F(a^*; z)$ be a solution of problem (2). The identity is known ^(6, 7)

$$\sum_{s=1}^r \lambda_s \tilde{F}^*(\tilde{A}; z_s) \equiv 0, \quad \lambda_s > 0, \quad s = 1, \dots, r, \quad (9)$$

to which correspond the two identities

$$\sum_{s=1}^r \lambda_s \tilde{F}^{*'}(\tilde{A}; z_s) \equiv 0, \quad \lambda_s > 0, \quad (10)$$

$$\sum_{s=1}^r \lambda_s \tilde{F}^{*''}(\tilde{A}; z_s) \equiv 0, \quad \lambda_s > 0, \quad (11)$$

with linear dependence in the narrow sense ⁽⁷⁾ among the real forms, where $\{z_s\}_1^r$ are the Chebyshev deviation points* ⁽⁷⁾.

* For brevity, deviation points are the points of maximum deviation $|F(a; z) - f(z)|$ on G .

To the identities (10), (11) there corresponds the matrix

$$\left\| \begin{array}{c} \varphi_j^{*'}(z_s) \\ \varphi_j^{*''}(z_s) \end{array} \right\|_{j=0,1,\dots,n; s=1,2,\dots,r}. \quad (12)$$

whose rank is not less than $r-1$ by virtue of the linear dependence in the narrow sense of the forms $\widehat{F}^{*'}(A; z_s)$ (or $\widehat{F}^{*''}(A; z_s)$).

Lemma 2. If the rank of the matrix (12) is equal to $r-1$, i.e.

$$\sum_{s=1}^r c_s \vec{\varphi}_s^* = 0,$$

$c_s \neq 0$, $s = 1, \dots, r$, then $c_s = \lambda_s k$, $k = \text{const}$.

Theorem 2. In order that an admissible polynomial $F(a^*; z)$ be least deviating from the function $f(z)$ on G , it is necessary and sufficient that, for some subsystem of deviation points $\{z_s\}_1^r$, one of the following two conditions be fulfilled:

- 1) in the case of linear independence of the vectors $\vec{\varphi}_s^*$, $s = 1, \dots, r$, corresponding to the chosen deviation points, for the numbers $K_s^{(t)}$, $K_s^{(t)*}$ from (3), (4) the following conditions must be satisfied:

- a) the rank m of the matrix

$$\left\| \begin{array}{c} K_s^{(t)} \\ K_s^{(t)*} \end{array} \right\|_{t=1,2,\dots,p; s=r+1,\dots,2n+2}, \quad (13)$$

corresponding to the added vectors d_s , is less than or equal to $2p-1$;

- b) the nonzero numbers

$$\left| \begin{array}{cccc} K_{l_1}^{(t)} & \dots & K_{l_{2p-1}}^{(t)} & K_s^{(t)} \\ K_{l_1}^{(t)*} & \dots & K_{l_{2p-1}}^{(t)*} & K_s^{(t)*} \end{array} \right|_{t=1,\dots,p; s=1,\dots,2n+2} \quad (14)$$

must be of one sign, where in (14) the first $2p-1$ columns are linearly independent and include m linearly independent columns of the matrix (13);

- 2) there exists a linear dependence in the narrow sense of the vectors $\vec{\varphi}_s^*$

$$\sum_{s=1}^r c_s \vec{\varphi}_s^* = 0, \quad (*)$$

where the numbers c_s , $s = 1, \dots, r$, are of one sign.

From 1) of Theorem 2 there follows the identity

$$\sum_{s=1}^r \lambda_s F^*(a; z_s) \equiv \sum_{j=0}^n \gamma_j a_j, \quad \lambda_s > 0, \quad \gamma_j = \sum_{t=1}^p \mu_t \alpha_j^{(t)}, \quad j = 0, \dots, n; \quad (15)$$

$$\lambda_s = \sum_{t=1}^p (\mu'_t K_s^{(t)} + \mu''_t K_s^{(t)*}), \quad s = 1, \dots, r,$$

and the numbers $\mu_t = \mu'_t + i\mu''_t$ are computed, in the notation of Theorem 2, from the system of equations

$$\sum_{t=1}^p (\mu'_t K_\nu^{(t)} + \mu''_t K_\nu^{(t)*}) = 0, \quad \nu = 1, \dots, 2p-1.$$

From (*) we obtain

$$\sum_{s=1}^r c_s F^*(a; z_s) \equiv 0, \quad c_s > 0. \quad (16)$$

A number of other theorems have been obtained, formulated with and without added vectors d_s , of the same type as in ^(11,12).

For the value of the best approximation ρ in case 1) of Theorem 2, from (15) we obtain

$$\rho = \left[\sum_{t=1}^p \mu_t \alpha_t - \sum_{s=1}^r \lambda_s \operatorname{sgn} \overline{\delta(z_s)} f(z_s) \right] / \sum_{s=1}^r \lambda_s \quad (17)$$

and in case 2) of Theorem 2, from (16) we obtain:

$$\rho = \left[- \sum_{s=1}^r c_s \operatorname{sgn} \overline{\delta(z_s)} f(z_s) \right] / \sum_{s=1}^r c_s. \quad (18)$$

Corollary. If $f(z) \equiv 0$ and in (1) $\sum |a_t| > 0$, then in Theorem 2 only case 1) occurs.

Remark 1. If $f(z)$, $\varphi_j(z)$, $j = 0, \dots, n$, are only bounded and the set G is of arbitrary nature, then the study of the problem reduces to the preceding one if one considers all distinct vectors

$$\vec{\varphi}^{(\nu)}(z_s) = [\varphi_0^{(\nu)}(z_s), \dots, \varphi_n^{(\nu)}(z_s), f^{(\nu)}(z_s)], \quad z_s \in G,$$

and takes their closure.

Remark 2. On the basis of the results obtained, a generalization to problem (2) has been obtained of the results of work ⁽⁸⁾ on the connection between Chebyshev approximations and the extremal moment problem.

2. Theorem 3. For any function $f(z)$ and polynomial $F(a; z)$ one can indicate p ($1 \leq p \leq n + 1$) linearly independent relations under which the polynomial $F(a; z)$ will be least-deviating from $f(z)$ on G .

Methods have been obtained for solving the inverse problem of V. A. Markov, analogous to those applied in the real domain ⁽¹²⁾, etc.

Let us consider one of the methods for solving the inverse problem. For a given function $f(z)$ and polynomial $F(a^*; z)$, $z \in G$, choose p linearly independent relations so that the points of maximal deviation $\{z_s\}_1^r$, for which the vectors φ_s^* , $s = 1, \dots, r$, are linearly independent, form a Chebyshev subsystem of points of deviation, i.e., so that under the chosen relations the polynomial $F(a^*; z)$ is a solution of problem (2).

To solve the problem, take numbers $\lambda_s > 0$, $s = 1, \dots, r$, and put

$$\sum_{s=1}^r \lambda_s \varphi_j^*(z_s) = \gamma_j, \quad j = 0, \dots, n.$$

Take numbers μ_t and p linearly independent vectors

$$\vec{\alpha}^{(t)} = [\alpha_0^{(t)}, \dots, \alpha_n^{(t)}], \quad t = 1, \dots, p,$$

satisfying the condition

$$\gamma_j = \sum_{t=1}^p \mu_t \alpha_j^{(t)}, \quad j = 0, \dots, n.$$

The sought relations will be

$$\sum_{j=0}^n a_j \alpha_j^{(t)} = \alpha_t, \quad t = 1, \dots, p,$$

where

$$\alpha_t = \sum_{j=0}^n a_j^* \alpha_j^{(t)}.$$

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