

BANACH SPACES WITH A $\|(J\|)$ -METRIC.

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1966

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Abstract

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UDC 513.88:517.948.35

MATHEMATICS

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BANACH SPACES WITH A J -METRIC.

J -NONNEGATIVE OPERATORS

(Presented by Academician L. S. Pontryagin on 5 XI 1965)

1. Let a Banach space $\mathfrak{B}$ be the direct sum of two of its subspaces: $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, and let $P_{\pm}$ be the bounded projectors corresponding to this decomposition ($\mathfrak{B}_{\pm} = P_{\pm}\mathfrak{B}$). Introduce in $\mathfrak{B}$ the so-called J -metric by putting

$$J(x) = \|P_+x\|^2 - \|P_-x\|^2 \quad (x \in \mathfrak{B}).$$

(We note that the squares here could be replaced by any other positive power; cf. ^(1,2).) The function $J(x)$ is, obviously, strongly continuous and, moreover, uniformly (strongly) continuous in every ball $\|x\| \leq C$. Depending on the sign assumed by this function on vectors and sets of vectors from $\mathfrak{B}$, these vectors and sets are naturally classified as J -positive (J -nonnegative), J -negative (J -nonpositive), and J -neutral.

For the case in which $\mathfrak{B}$ is a Hilbert space and $P_{\pm}$ are orthoprojectors ($\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$), the geometry and the theory of operators in spaces $\mathfrak{B}$ with a J -metric were first studied by L. S. Pontryagin, M. G. Krein, and the author under the assumption $\min\{\dim \mathfrak{B}_+, \dim \mathfrak{B}_-\} = \varkappa < \infty$ ⁽³⁻⁸⁾, and then (already without the restriction $\varkappa < \infty$, partially removed earlier in ⁽⁶⁾, §13) this theory received further development and application in a number of works by Soviet and foreign authors (for a complete bibliography see ^(9,10)). The passage to a Banach space, apparently, was first undertaken by F. Bonsall ⁽¹⁾ ($\varkappa = 1$), then by M. L. Brodskii ⁽¹¹⁾ and K. Fan ⁽²⁾ ($\varkappa < \infty$), and for $\varkappa = \infty$ by M. G. Krein ⁽¹²⁾, the author ⁽¹³⁾, and K. Fan ⁽¹⁴⁾.

2. A J -nonnegative (J -nonpositive) subspace is called **maximal** if it is not a proper part of any other J -nonnegative (J -nonpositive) subspace of $\mathfrak{B}$. From Zorn's lemma one obtains (cf. ⁽⁹⁾, §3, 10) the so-called

Maximality principle. *Every J -nonnegative (J -nonpositive) subspace is contained in some maximal J -nonnegative (J -nonpositive) subspace.*

The following facts hold (cf. ⁽⁹⁾, §6):

1°. The projector P_+ (P_-) maps every J -nonnegative (J -nonpositive) linear manifold $\mathfrak{L}$ from $\mathfrak{B}$ homeomorphically onto the linear manifold $P_+\mathfrak{L} \subset \mathfrak{B}_+$ ($P_-\mathfrak{L} \subset \mathfrak{B}_-$). In particular, $\dim \mathfrak{L} \leq \dim \mathfrak{B}_+$ ($\dim \mathfrak{L} \leq \dim \mathfrak{B}_-$).

2°. If, under the conditions of assertion 1°, $P_+\mathfrak{L} = \mathfrak{B}_+$ ($P_-\mathfrak{L} = \mathfrak{B}_-$), then $\mathfrak{L}$ is a maximal J -nonnegative (J -nonpositive) subspace.

Denote by $\mathfrak{T}_+$ ($\mathfrak{T}_-$) the class of all subspaces $\mathfrak{L}$ of $\mathfrak{B}$ that are mapped by the projector P_+ (P_-) one-to-one onto all of $\mathfrak{B}_+$ ($\mathfrak{B}_-$). With the aid of the Hahn-Banach theorem the following assertion is proved:

3°. Every one-dimensional J -nonnegative (J -nonpositive) subspace is contained in some maximal J -nonnegative (J -nonpositive) subspace $\mathfrak{L} \in \mathfrak{T}_+$ ($\mathfrak{L} \in \mathfrak{T}_-$).

For subspaces of larger dimension this assertion is, generally speaking, no longer true.

Lemma 1. In order that every J -nonnegative subspace be contained in some maximal J -nonnegative subspace of the class $\mathfrak{T}_+$, it is necessary and sufficient that every bounded linear operator acting from an arbitrary subspace $\mathfrak{B}_+^0 \subset \mathfrak{B}_+$ into the subspace $\mathfrak{B}_-$ admit an extension, with the same norm, to an operator acting from all of $\mathfrak{B}_+$ into $\mathfrak{B}_-$.

Remark. As is known ^(15,16), the condition of Lemma 1 will be satisfied if, for example, any one of the following requirements is satisfied:

α) $\mathfrak{B}_+$ is a unitary space; β) $\mathfrak{B}_-$ is a space of type $\mathfrak{M}$.

An assertion analogous to Lemma 1 holds, of course, also for J -nonpositive subspaces.

A special role is played by the particular case in which

$$\min\{\dim \mathfrak{B}_+, \dim \mathfrak{B}_-\} = \varkappa < \infty.$$

For definiteness we shall assume $\varkappa = \dim \mathfrak{B}_+$. In the following lemma $0 \leq \varkappa < \infty$.

Lemma 2. If the linear manifold $\mathfrak{L}(\subset \mathfrak{B})$ contains a $\varkappa$ -dimensional J -positive subspace $*$, then the same property is possessed by every linear manifold $\mathfrak{D}(\subset \mathfrak{L})$ dense in $\mathfrak{L}$ (i.e. such that $\overline{\mathfrak{D}} \supset \mathfrak{L}$).

3. A linear operator V , defined on some linear manifold $\mathfrak{D}_V$ in $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, will be called J -nonnegative if from $x \in \mathfrak{D}_V$ and $J(x) \geq 0$ it always follows that $J(Vx) \geq 0$. J -nonpositive operators are defined analogously; for them in what follows all assertions are easily obtained from those given in the text.

Theorem 1. Let V be a J -nonnegative operator defined on some linear manifold $\mathfrak{D}_V$ in $\mathfrak{B}$, and let $\mathfrak{D}_V \cap \mathfrak{B}_+ \neq \{0\}$. The operator V is bounded if and only if the operator P_+V is bounded.

From this theorem and Lemma 2 one can obtain the following generalization and strengthening of one result of M. L. Brodskii (see ⁽¹¹⁾, 1°):

Theorem 2. Let V be a J -nonnegative operator in $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, and let $\dim \mathfrak{B}_+ = \kappa$ ($1 \leq \kappa < \infty$). If $\mathfrak{D}_V \cap \mathfrak{B}_+ \neq \{0\}$ and $\mathfrak{D}_V$ contains a κ -dimensional J -positive subspace, while the operator V annihilates no J -positive vector, then this operator is bounded.

In the particular case when $\mathfrak{B}$ is a Hilbert space and $\mathfrak{B}_\pm$ are its mutually orthogonal subspaces, the conditions of Theorem 2 imposed on $\mathfrak{D}_V$ are satisfied, for example, when $\mathfrak{D}_V = \mathfrak{B}$. The condition requiring that the operator V not annihilate J -positive vectors is essential, since even in a Hilbert space $\mathfrak{B}$, for $\mathfrak{D}_V = \mathfrak{B}$, one can give examples of one-dimensional J -nonnegative operators which, when this condition is violated, turn out to be unbounded.

Theorem 3. If, in a Banach space $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, a J -nonnegative operator V annihilates some J -positive vector, then the linear manifold $\mathfrak{R}_V = V\mathfrak{D}_V$ is J -nonnegative (and, consequently, by 1° , $\dim \mathfrak{R}_V \leq \dim \mathfrak{B}_+$) (cf. $(^{11}), 1^\circ$).

Corollary. If, under the conditions of Theorem 3, $\mathfrak{D}_V = \mathfrak{B}$, then the linear manifold $\mathfrak{R}_V$ is a J -nonnegative invariant linear manifold for the operator V .

4. The question touched upon in the corollary to Theorem 3, concerning the existence for a J -nonnegative operator of a J -nonnegative invariant subspace, is connected with a well-known problem in operator theory which has given rise to an extensive literature $(^{1-8,10-14,17})$. The central point in these works is the proof of the existence of maximal J -nonnegative invariant subspaces of the class $\mathfrak{T}_+$ for various classes of operators in spaces with a J -metric. The most fruitful idea here proved to be that of M. G. Krein $(^{4-6,12})$ [[unclear: continuation cut off at bottom]]

* A linear manifold (subspace) will be called J -positive if all its nonzero vectors are J -positive. J -negative linear manifolds are defined analogously.

...of obtaining such proofs by means of various fixed-point principles.

The greatest progress in the question under consideration has been achieved in the paper $(^{12})$ (see also $(^{13})^*$, where, in proving the existence of J -nonnegative invariant subspaces of class $\mathfrak{J}_+$ for a J -nonnegative operator V with $\mathfrak{D}_V = \mathfrak{B}$, the following requirements are imposed: 1) V is bounded; 2) the operator P_+VP_- admits approximation in norm by finite-dimensional operators (in the case when $\mathfrak{B}$ is a Hilbert space or a Banach space with a basis, this is equivalent to the requirement that P_+VP_- be a completely continuous operator).

The assertion contained in a recent paper of K. Fan $(^{14})$, Corollary 2 of Theorem 2), if valid, would constitute, in the case of its applicability, a further step toward the generalization of the results named in $(^{9,12,13,17})$, since condition 2) is absent from the formulation of this assertion of K. Fan. However, unfortunately, the proof of Corollary 2 in $(^{14})$, as its analysis shows, contains a fundamental error.

Thus, even for J -unitary operators V ($\mathfrak{D}_V = \mathfrak{R}_V = \mathfrak{B}$, $J(Vx) = J(x)$, $x \in \mathfrak{B}$) in a Hilbert space $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$, the possibility of dispensing with condition

2) remains problematic (condition 1) is satisfied automatically in this case ⁽¹⁸⁾). Closely connected with this question is another unsolved problem, already for families of commuting J -unitary operators, posed by R. S. Phillips ⁽¹⁹⁾ (see the commentary to the translation ⁽²⁰⁾ of this article).

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Received
3 XI 1965

CITED LITERATURE

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* We take this opportunity to make the following clarification: in Theorem 3 of ⁽¹³⁾, if $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$ is a Banach space, then one must additionally require of the operator A that it send every J -nonnegative subspace of class $\mathfrak{J}_+$ into a subspace of class $\mathfrak{J}_+$. In a Hilbert space $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$ this requirement is satisfied automatically by virtue of the maximality principle (see item 2) and item a) of the remark to Lemma 1 of the present note.

Note: Figure translations are in progress. See original paper for figures.

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