

PROOF OF THE UPPER SEMICONINUITY OF THE VARIATION OF A SET

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Abstract

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MATHEMATICS

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PROOF OF THE UPPER SEMICONTINUITY OF THE VARIATION OF A SET

(Presented by Academician A. N. Kolmogorov on 15 VI 1965)

In this note it is proved that the k -dimensional variation $V_k(e)$ of a closed set e , situated in Euclidean space, is upper semicontinuous if the variations of smaller dimension are bounded. Let us recall the definition of k -dimensional variation (see ⁽¹⁾).

Let E_n be n -dimensional Euclidean space; Ω_n^k the space of $(n-k)$ -dimensional planes β_{n-k} in E_n ; μ_n^k the Haar measure in the space Ω_n^k , invariant with respect to the group of transformations of the space Ω_n^k generated by motions of the space E_n ; $V_0(e)$ the number of components of the set e . The k -th variation of a closed set e is the number

$$V_k(e) = \int_{\Omega_n^k} V_0(e \cap \beta_{n-k}) d\mu_n^k.$$

The normalization of the measure μ_n^k is chosen so that the k -th variation of a k -dimensional unit cube is equal to 1.

In the space of subsets of E_n we shall use the deviation metric, i.e., as the distance between sets e and f from E_n we take the quantity $r(e, f) = \max[\rho(e, f), \rho(f, e)]$, where $\rho(A, B) = \sup_{x \in A} \rho(x, B)$, and $\rho(x, B)$ is the distance from the point x to the set B .

After these explanations we formulate precisely our result.

Theorem 1. If, for closed sets e and $e_1, e_2, \dots, e_i, \dots$,

- 1) $\lim_{i \rightarrow \infty} r(e, e_i) = 0$;
- 2) all $V_m(e_i)$ ($m = 0, 1, 2, \dots, (k-1)$; $i = 1, 2, \dots$) do not exceed some constant M ,

then

$$\sup_{i \rightarrow \infty} V_k(e_i) \geq V_k(e).$$

Consider the minimal σ -algebra $\nu_{k,n}$ of sets in E_n containing all closed subsets of E_n with finite variations whose dimension is not greater than k .

The variation $V_k(e)$, extended to all sets $e \in \nu_{k,n}$, turns out to be a countably additive measure of order k , which is semicontinuous (see the formulation of the theorem) and on all polyhedra coincides with k -dimensional Lebesgue measure. It seems plausible that a measure satisfying the listed conditions is unique, i.e., coincides with $V_k(e)$.

Notation.

$$d(g) = \sup_{a \in g, b \in g} \rho(a, b)$$

is the diameter of the set g ; $\partial(e)$ is the boundary of the set e ; the depth of immersion of the set e in $g \cap \beta$ will be called

$$r(e, g, \beta) = \sup \rho(x, \partial(g \cap \beta)),$$

where the least upper bound is taken over all $x \in e \cap g \cap \beta$;

$$d(e, g, \beta_p) = \{\text{mes}_p(g \cap \beta_p) - \text{mes}_p(e \cap g \cap \beta_p)\}^{1/p},$$

where $\beta_p \subset E_n$ is a p -dimensional plane;

$$m(e, g, \beta) = \min\{\Gamma(e, g, \beta), [d(e, g, \beta)]\},$$

$$k(g, \beta) = \Gamma(g, g, \beta)[d(g)]^{-1}.$$

Lemma 1. Let there be given a convex domain $g \subset E_n$ such that $k(g, E_n) \geq \gamma > 0$, and a set $e \subset E_n$ such that every component of the set $e \cap g$ meets $\partial(g \cap E_n)$.

Denote by $\Omega(\Delta_1, \Delta_2, g)$ the set of hyperplanes β_{n-1} determined by the conditions: $k(g, \beta_{n-1}) \geq \Delta_1$ and $m(e, g, \beta_{n-1}) \geq \Delta_2 m(e, g, E_n)$.

There exist numbers $\varepsilon > 0$, $\Delta_1 > 0$, and $\Delta_2 > 0$, depending only on n and γ , such that

$$\mu_n^1[\Omega(\Delta_1, \Delta_2, g)] \geq \varepsilon m(e, g, E_n).$$

Lemma 2. Let the set $e \subset E_n$ be closed and bounded, and let $g_1, g_2, \dots, g_p$ be convex, pairwise disjoint sets, with

$$k(g_i, E_n) \geq \gamma > 0 \quad (i = 1, 2, \dots, p).$$

Then for every $k = 0, 1, 2, \dots, n-1$,

$$\mathbf{V}_k(e) \geq \sum_{i=1}^p C_{i,k} [m(e, g_i, E_n)]^k,$$

where $\{C_{i,k}\}$ are nonnegative numbers such that, for every i ,

$$\sum_{k=0}^{n-1} C_{i,k} \geq C(n, \gamma) > 0$$

($C(n, \gamma)$ depends only on n and γ).

Proof. The lemma is proved by induction on n .

Consider the case $n = 1$. Among all intervals $\{g_i\}$ intersecting one and the same component of the set e , at most two intervals can satisfy the condition $m(e, g_i, E_1) > 0$, and therefore

$$\mathbf{V}_0(e) \geq \sum_{i=1}^p [m(e, g_i, E_1)]^0 \cdot \frac{1}{2}.$$

(Here it is assumed that if $m(e, g, E_1) = 0$, then $[m(e, g_i, E_1)]^0 = 0$); that is, for $n = 1$ the lemma is proved.

Now consider the general case. Recall that

$$\mathbf{V}_k(e) = C(n, k) \int_{\Omega_n^1} \mathbf{V}_{k-1}(e \cap \beta_{n-1}) d\mu_n^1,$$

where $C(n, k) > 0$ is a normalizing constant independent of e . By the induction hypothesis, for every plane β_{n-1} we have

$$\mathbf{V}_{k-1}(e \cap \beta_{n-1}) \geq \sum_i C_{i,k-1}(\beta_{n-1}) [m(e, g_i, \beta_{n-1})]^{k-1},$$

$$\sum_{q=1}^{n-2} C_{i,q}(\beta_{n-1}) \geq C((n-1), \gamma) \quad (i = 1, 2, \dots, p).$$

Here we assume that if $g_i \cap \beta_{n-1}$ is empty, then $m(e, g_i, \beta_{n-1}) = 0$.

Integrating the last inequality, we obtain

$$\frac{\mathbf{V}_k(e)}{C(n, k)} \int_{\Omega_n^1} \mathbf{V}_{k-1}(e \cap \beta_{n-1}) d\mu_n^1 \geq \sum_i \int_{\Omega_n^1} C_{i,k-1}(\beta_{n-1}) [m(e, g_i, \beta_{n-1})]^{k-1} d\mu_n^1.$$

Put

$$C_{i,k} = C(n,k)[m(e, g_i, E_n)]^{-k} \int_{\Omega_n^1} C_{i,k-1}(\beta_{n-1})[m(e, g_i, \beta_{n-1})]^{k-1} d\mu_n^1$$

$$(k = 1, 2, \dots, n-1); \quad C_{i,0} = 1, \quad \text{if } g_i \cap e \text{ contains a component meeting}$$

$\partial(g_i, E_n)$ components, and $C_{i,0} = 0$ otherwise; we have

$$V_0(e) \geq \sum_i C_{i,0} = \sum_i C_{i,0} [m(e, g_i, E_n)]^0.$$

Thus, for every $k = 0, 1, \dots, n-1$,

$$V_k(e) \geq \sum_i C_{i,k} [m(e, g_i, E_n)]^{k-1},$$

and, by virtue of Lemma 1,

$$\begin{aligned} \sum_{k=0}^{n-1} C_{i,k} &= C_{i,0} + \sum_{k=1}^{n-1} C(n,k) \int_{\Omega(\Delta_1, \Delta_2, g_i)} C_{i,k-1}(\beta_{n-1}) [m(e, g_i, \beta_{n-1})]^{k-1} \times \\ &\quad \times [m(e, g_i, E_n)]^{-k} d\mu_n^1 \geq C_{i,0} + \sum_{k=1}^{n-1} C(n,k) [m(e, g_i, E_n)]^{-k} \times \\ &\quad \times \int_{\Omega(\Delta_1, \Delta_2, g_i)} C_{i,k-1}(\beta_{n-1}) [\Delta_2 m(e, g_i, E_n)]^{k-1} d\mu_n^1 \geq \\ &\geq C_{i,0} + \sum_{k=1}^{n-1} C(n,k) \Delta_2^{k-1} [m(e, g_i, E_n)]^{-1} \int_{\Omega(\Delta_1, \Delta_2, g_i)} C_{i,k}(\beta_{n-1}) d\mu_n^1 \geq \\ &\geq C_{i,0} + \min_k C(n,k) \Delta_2^{k-1} [m(e, g_i, E_n)]^{-1} \int_{\Omega(\Delta_1, \Delta_2, g_i)} \sum_{k=1}^{n-k} C_{i,k}(\beta_{n-1}) d\mu_n^1 \geq \\ &\geq C_{i,0} + C'(n, \Delta_1) [m(e, g_i, E_n)]^{-1} \varepsilon \cdot m(e, g_i, E_n) C((n-1), \Delta_1, \gamma) \geq C(n, \gamma). \end{aligned}$$

The lemma is proved.

Lemma 3. Let the closed set e be an ε -net for the measurable set $f \subset E_n$. Then, for every $\varepsilon \leq \varepsilon(f)$, for some $k = k(\varepsilon)$ the inequality

$$V_k(e) \geq C(n) [\text{mes}_n f - \text{mes}_n e] \varepsilon^{k-n},$$

holds, where $C(n)$ depends only on n .

Proof. We shall assume that $\Delta = \text{mes}_n f - \text{mes}_n e > 0$. Fix p equal pairwise nonintersecting cubes $g_1, g_2, \dots, g_p$ with side 4ε and such that

$$\sum_i \text{mes}_n(f \cap g_i) \geq (1 - \frac{1}{8}\Delta) \text{mes}_n f.$$

Since

$$\sum_i [m(e, g_i, E_n)]^n \geq C_1(n)\Delta,$$

then by virtue of Lemma 2 one can indicate indices $i_1, i_2, \dots, i_q$ and a number k such that

$$V_k(e) \geq \frac{C(n, \gamma)}{n} \sum_{s=1}^q [m(e, g_{i_s}, E_n)]^k,$$

$$\sum_{s=1}^q m(e, g_{i_s}, E_n)^n \geq C_2(n)\Delta,$$

and consequently,

$$V_k(e) \geq C(n) \cdot \Delta \cdot \varepsilon^{k-n}.$$

The lemma is proved.

Notation.

Let φ_n^k be the bundle of k -dimensional planes τ_k passing through one and the same point; let $\overline{m}_n^k$ be the measure in the space φ_n^k , invariant with respect to the group of transformations generated by rotations of space; $\beta_{s+n-k}(\beta_s, \tau_k)$ be the plane of dimension $s+n-k$ containing the s -dimensional plane $\beta_s \subseteq \tau_k$ and the $(n-k)$ -dimensional plane $\beta_{n-k} \subset E_n$, which is ...

orthogonal complement to τ_k

$$V_{s, \tau_k}(e) = \int_{\beta_s \subset \tau_k} V_0(e \cap \beta_{s+n-k}(\beta_s, \tau_k)) d\mu_k^{k-s}.$$

Lemma 4. Let the set $f \subset E_n$ be closed and such that, for every plane $\beta_s \subset \tau_k$ (τ_k fixed) and for every pair of components f_1 and f_2 of the set $f \cap \beta_{s+n-k}(\beta_s, \tau_k)$,

$$\min_{\substack{a \in f_1 \\ b \in f_2}} \rho(a, b) \geq \Delta > 0$$

(Δ does not depend on β_s, f_1, f_2), and let the set e be an ε -net for f and lie in the $(\frac{1}{4}\Delta)$ -neighborhood of the set f . Then, if $\varepsilon < \varepsilon(f)$ ($\varepsilon(f)$ does not depend on e), then

$$\sum_{s=0}^{k-1} V_{s, \tau_s}(e) \geq C(k) [V_{k, \tau_k}(f) - V_{k, \tau_k}(e)] \varepsilon^{-1}$$

($C(k)$ —see Lemma 3).

Lemma 5. If a closed set is equal to $\lim_{i \rightarrow \infty} e_i$, then for every plane τ_k

$$\lim_{i \rightarrow \infty} \sum_{s=0}^{k-1} V_{s, \tau_k}(e_i) = +\infty,$$

provided only that

$$\inf_{i \rightarrow \infty} V_{k, \tau_k}(e_i) < V_{k, \tau_k}(e).$$

The lemma follows easily from the preceding lemma, and from it the proof of Theorem 1 is obtained, since

$$V_s(f) = C(n, k, s) \int_{\varphi_n^k} V_{s, \tau_k}(e) dm_n^k,$$

where $C(n, k, s)$ is a normalizing factor independent of the set f .

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1. A. G. Vitushkin, *On multidimensional variations*, Moscow, 1955.

Note: Figure translations are in progress. See original paper for figures.

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