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Abstract

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MATHEMATICS

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ON THE BIRTH OF PERIODIC MOTION FROM A TRAJECTORY ISSUING FROM AN EQUILIBRIUM STATE OF SADDLE-SADDLE TYPE AND RETURNING TO IT

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1. Consider a system of differential equations of order $(m + n + 1)$

$$\begin{aligned} dx/dt &= P(x, y, z, \mu), & dy/dt &= Q(x, y, z, \mu), \\ dz/dt &= R(x, y, z, \mu), \end{aligned} \quad (1)$$

where $P(P_1, P_2, \dots, P_m)$, $Q(Q_1, Q_2, \dots, Q_n)$, and R are analytic or sufficiently smooth functions of the variables $x(x_1, x_2, \dots, x_m)$, $y(y_1, y_2, \dots, y_n)$, z , and of the parameter μ in some domain $G \times (-\mu_0, \mu_0)$. Suppose that for $\mu = 0$ system (1) has an equilibrium state $O \in G$, at which the characteristic equation has m roots with negative real parts, n roots with positive real parts, and one zero root.

By means of a linear nonsingular change of variables, system (1) can be brought to the form

$$\begin{aligned} dx/dt &= Ax + P_0(x, y, z, \mu), & dy/dt &= By + Q_0(x, y, z, \mu), \\ dz/dt &= R_0(x, y, z, \mu), \end{aligned} \quad (2)$$

where P_0, Q_0, R_0 , for $\mu = 0$, vanish at $O(0, 0, 0)$ together with their first derivatives; A is a matrix of order m , whose characteristic polynomial has roots $\lambda_1, \lambda_2, \dots, \lambda_m$ with negative real parts, and B is a matrix of order n , whose characteristic polynomial has roots $\gamma_1, \gamma_2, \dots, \gamma_n$ with positive real parts. Let

$$x = u(z), \quad y = v(z)$$

be a solution of the system

$$Ax + P_0(x, y, z, 0) = 0, \quad By + Q_0(x, y, z, 0) = 0$$

for sufficiently small z . Suppose that in the expansion

$$R_0(u(z), v(z), z, 0) = l_k z^k + l_{k+1} z^{k+1} + \dots \quad (3)$$

the first nonzero quantity l_k has an even index and, for definiteness, is positive.

The behavior of phase trajectories in a neighborhood of such a complex equilibrium state was studied in the work of R. M. Mintz ⁽¹⁾; she called the equilibrium state itself an equilibrium state of type C . This equilibrium state is formed by the coalescence of $2k$ simple equilibrium states of saddle type; therefore, it seems to us more natural to call O a complex equilibrium state of saddle-saddle type.

Let us describe the behavior of integral curves in a neighborhood of O . It is characterized first of all by the existence of an $(n + m)$ -dimensional separatrix surface π , passing through O and tangent to the x - and y -axes. This surface divides a neighborhood of O into two regions. In the region containing the half-axis $z \leq 0$, there exists a unique $(m + 1)$ -dimensional integral O^+ -semisurface (a semisurface consisting of O^+ -curves), which we shall denote by π^+ . In the region containing the half-axis $z \geq 0$, there exists a unique $(n + 1)$ -dimensional integral O^- -semisurface (a semisurface consisting of O^- -curves), which we shall denote by π^- . The boundaries of the semisurfaces π^+ and π^- , which we shall denote by Π^+ and Π^- , are manifolds of dimensions m and n , respectively. They lie in π and intersect at the point O . All integral curves, except those that lie in π^+ and π^- , pass by O .

The purpose of the present paper is to prove the following theorem.

Theorem. *Suppose that for $\mu = 0$ system (1) has the following properties: 1) there exists a trajectory Γ_0 issuing from O and returning to it again as $t \rightarrow +\infty$; 2) Γ_0 does not lie in Π^+ and Π^- ; 3) π^+ and π^- intersect along Γ_0 , without being tangent.**

Then, for all sufficiently small $\mu \neq 0$ for which the composite equilibrium state of saddle-saddle type disappears, a unique periodic motion of saddle type is born from Γ_0 .

The conditions for the disappearance of an equilibrium state of saddle-saddle type are completely analogous to the conditions for the disappearance of a composite equilibrium state of saddle-node type ⁽⁴⁾.

The proof of the theorem is based on the method of point mappings and consists in proving the existence of a fixed point of a mapping T of the plane $z = -d$ ($d > 0$) into itself.

We shall seek the mapping T in the form $T = T_1 T_0$, where T_0 is the mapping of the plane $z = -d$ into $z = d$ in a neighborhood of the origin, and T_1 is the mapping of $z = +d$ into $z = -d$ in a neighborhood of the global arc Γ_0 .

2. Construction of the mapping T_0 . Let us first consider system (2) for $\mu = 0$. As follows from the results of V. A. Pliss ^(2,3), in a neighborhood of the origin O there exist two integral surfaces

$$y = \psi(x, z), \quad x = \varphi(y, z), \quad (4)$$

whose dimensions are $m + 1$ and $n + 1$, respectively. The functions ψ and φ are sufficiently smooth and vanish at the origin together with their first derivatives. On these surfaces O will be a composite equilibrium state of saddle-node type. From the uniqueness of the integral O^+ -semisurface π^+ and the O^- -semisurface π^- it follows that, in the region to the left of π , the surface $y = \psi(x, z)$ coincides with π^+ , while in the region to the left of π the surface $x = \varphi(y, z)$ coincides with π^- .

By means of a certain sufficiently smooth change of variables in a sufficiently small neighborhood of the origin, system (2) can be written in the form

$$\begin{aligned} d\xi/dt &= A\xi + P'(\xi, \eta, z)\xi + \mu\alpha(z, \mu)R_0(z, \mu), \\ d\eta/dt &= B\eta + Q'(\xi, \eta, z)\eta + \mu\beta(z, \mu)R_0(z, \mu), \\ dz/dt &= R_0(z, \mu) + R'_1(\xi, \eta, z, \mu)\xi + R'_2(\xi, \eta, z, \mu)\eta, \end{aligned} \quad (5)$$

where P' , Q' , R'_1 , R'_2 are small for sufficiently small ξ, η, z, μ , while $R_0(z, 0) = l_k z^k + \dots$. In the new variables the equations of the surfaces (4) will be, respectively, $\eta = 0$ and $\xi = 0$, and the equations of Γ_0 will be $\xi = 0$, $\eta = 0$. It is obvious that the condition for the disappearance of the saddle-saddle will consist in the fulfillment of the inequality $R_0(z, \mu) > 0$. Suppose, for definiteness, that $(0, \bar{\mu}_0)$ is an interval of values of μ for which this inequality is satisfied. A value of μ from this interval will be denoted by $\bar{\mu}$.

* From the works of V. A. Pliss ^(2,3) it follows that π^+ and π^- are smooth surfaces.

Introduce into consideration the closed surface $S_{\bar{\mu}}$, consisting of the surfaces

$$\begin{aligned} V \equiv \|\xi\|^2 - R_0^2(z, \bar{\mu}) &= 0, & W \equiv \|\eta\|^2 - R_0^2(z, \bar{\mu}) &= 0, \\ S_0 : z = -d, & & S_1 : z = d. \end{aligned} \quad (6)$$

It is easy to see that inside $S_{\bar{\mu}}$, for sufficiently small d and sufficiently small μ , the condition $dz/dt > 0$ is satisfied. After eliminating the time t , system (5) inside $S_{\bar{\mu}}$ can be written in the form

$$\begin{aligned}\frac{d\xi}{dz} &= \frac{A\xi + P_1(\xi, \eta, z, \bar{\mu})\xi + \mu\alpha(z, \bar{\mu})R_0(z, \bar{\mu})}{R_0(z, \bar{\mu})}, \\ \frac{d\eta}{dz} &= \frac{B\eta + Q_1(\xi, \eta, z, \bar{\mu})\eta + \mu\beta(z, \bar{\mu})R_0(z, \bar{\mu})}{R_0(z, \bar{\mu})},\end{aligned}$$

where P_1 and Q_1 are small inside $S_{\bar{\mu}}$ for sufficiently small d and $\bar{\mu}_0$.

Lemma 1. For every pair ξ_0 and η_1 satisfying the conditions

$$\|\xi_0\| < a < R_0(-d, 0), \quad \|\eta_1\| < b < R_0(d, 0),$$

there exists a solution of system (7)

$$\xi(z) = \xi(z, \xi_0, \eta_1, \bar{\mu}), \quad \eta(z) = \eta(z, \xi_0, \eta_1, \bar{\mu}), \quad (8)$$

lying inside $S_{\bar{\mu}}$ and satisfying the conditions

$$\xi(-d) = \xi_0, \quad \eta(d) = \eta_1,$$

and moreover

$$\begin{aligned}\|\xi(z)\| &< \|\xi_0\| \exp \left[\lambda \int_{-d}^z \frac{dz}{R_0(z, \bar{\mu})} \right] + \bar{K}\mu, \\ \|\eta(z)\| &< \|\eta_1\| \exp \left[\gamma \int_{+d}^z \frac{dz}{R_0(z, \bar{\mu})} \right] + \bar{K}\mu,\end{aligned} \quad (9)$$

where

$$\max_{1 \leq i \leq m} \operatorname{Re} \lambda_i < \lambda < 0, \quad 0 < \gamma < \min_{1 \leq i \leq n} \operatorname{Re} \gamma_i,$$

and $\bar{K}$ is some fixed constant.

Lemma 2. The estimates

$$\begin{aligned}\left\| \frac{\partial \xi}{\partial \xi_0} \right\| + \left\| \frac{\partial \eta}{\partial \xi_0} \right\| &< \exp \left[\lambda \int_{-d}^z \frac{dz}{R_0(z, \bar{\mu})} \right], \\ \left\| \frac{\partial \xi}{\partial \eta_1} \right\| + \left\| \frac{\partial \eta}{\partial \eta_1} \right\| &< \exp \left[\gamma \int_{+d}^z \frac{dz}{R_0(z, \bar{\mu})} \right]\end{aligned}$$

hold for all $z \in [-d, d]$ for sufficiently small d and $\bar{\mu}_0$.

Consider the integral curve (8). Obviously, specifying (ξ_0, η_1) makes it possible to determine uniquely

$$\xi_1 = \xi(d, \xi_0, \eta_1, \bar{\mu}) \quad \text{and} \quad \eta_0 = \eta(-d, \xi_0, \eta_1, \bar{\mu}).$$

The mapping of the points $M_0(\xi_0, \eta_0) \in S_0$ into the points $M_1(\xi_1, \eta_1) \in S_1$ obtained in this way will be denoted by T_0 . At the same time, note that, since

$$\lim_{\bar{\mu} \rightarrow 0} \int_{-d}^d \frac{dz}{R_0(z, \bar{\mu})} = +\infty,$$

we have

$$\|\xi_1\| + \|\eta_0\| + \|\partial \xi_1 / \partial \xi_0\| + \|\partial \eta_0 / \partial \xi_0\| + \|\partial \xi_1 / \partial \eta_1\| + \|\partial \eta_0 / \partial \eta_1\| \rightarrow 0 \quad (10)$$

as $\bar{\mu} \rightarrow 0$.

3. Construction of the mapping T_1 . In the variables ξ, η, z , Γ_0 intersects S_0 and S_1 at the points $M_0^0(0, 0)$ and $M_1^1(0, 0)$. As is known, for sufficiently small

$$0 < a < \frac{1}{2}R_0(-d, 0)$$

one can indicate such a

$$0 < b < \frac{1}{2}R_0(d, 0)$$

that, for all sufficiently small μ , the trajectories passing through the points $M_1 \in S_1$, lying in the b -neighborhood of M_1^1 , will intersect S_0 at points $\bar{M}_0(\bar{\xi}_0, \bar{\eta}_0)$, lying in the a -neighborhood of M_0^0 . The resulting map-

We shall denote the mapping by T_1 . In the variables under consideration it will be written in the form

$$\begin{aligned} \bar{\xi}_0 &= f_1(\xi_1, \eta_1, \mu) = A_1(\mu) + A_{11}(\mu)\xi_1 + A_{12}(\mu)\eta_1 + \dots, \\ \bar{\eta}_0 &= g_1(\xi_1, \eta_1, \mu) = A_2(\mu) + A_{21}(\mu)\xi_1 + A_{22}(\mu)\eta_1 + \dots \end{aligned} \quad (11)$$

Since, by assumption, π^+ and π^- intersect along Γ_0 without tangency, $|A_{22}(0)| \neq 0$.

4. The completion of the proof of the theorem is reduced to proving the existence for the mapping $T = T_1 T_0$ of a fixed point $M^*(\xi_0^*, \eta_0^*)$, tending to M_0^0 as $\mu \rightarrow 0$. Denote by $\sigma(a, b)$ the image of the domain $\|\xi_0\| < a, \|\eta_1\| < b$ under the mapping $(\xi_0, \eta_1) \rightarrow (\xi_0, \eta_0)$, where $\eta_0 = \eta(-d, \xi_0, \eta_1, \mu)$. It is easy to see, by virtue of the choice of a and b , that the mapping T is defined on $\sigma(a, b) \in S_0$ and maps $\sigma(a, b)$ into S_0 .

Let us consider the equations for finding the coordinates of a fixed point. In the variables ξ_0^*, η_1^* they will be written in the form

$$\begin{aligned} F(\xi_0^*, \eta_1^*, \bar{\mu}) &\equiv \xi_0^* - f_1(\xi(d, \xi_0^*, \eta_1^*, \bar{\mu}), \eta_1^*, \bar{\mu}) = 0, \\ G(\xi_0^*, \eta_1^*, \bar{\mu}) &= \eta(-d_0, \xi_0^*, \eta_1^*, \bar{\mu}) - g_1(\xi(d, \xi_0^*, \eta_1^*, \bar{\mu}), \eta_1^*, \bar{\mu}) = 0. \end{aligned} \quad (12)$$

Since, using (10), F and G can be extended continuously to the boundary $\bar{\mu} = 0$, together with their derivatives, and since for $\bar{\mu} = 0$ the system under consideration has the unique solution $\xi_0^* = 0, \eta_1^* = 0$, with Jacobian, at $\xi_0^* = \eta_1^* = \bar{\mu} = 0$, equal to $-|A_{22}(0)|$, it follows that for sufficiently small $\bar{\mu} > 0$ we obtain that the system (12) has a unique solution $\xi_0^*(\bar{\mu}), \eta_1^*(\bar{\mu})$, where $\xi_0^*(\bar{\mu})$ and $\eta_1^*(\bar{\mu})$ tend to zero together with $\bar{\mu}$. Consequently, the mapping T has a fixed point $M^*(\xi_0^*, \eta_0^*)$, tending to M_0^0 as $\bar{\mu} \rightarrow 0$. We shall prove that M^* is a fixed point of saddle type. Obviously, the characteristic equation, using the estimates (10), can be written in the form

$$\alpha_n(\bar{\mu})z^{m+n} + \dots + \alpha_1(\bar{\mu})z^{m+1} + (|A_{22}(0)| + O(\bar{\mu}))z^m + \beta_m(\bar{\mu})z^{m-1} + \dots + \beta_1(\bar{\mu}) = 0, \quad (13)$$

where $\alpha_i(\bar{\mu})$ and $\beta_i(\bar{\mu})$ tend to zero as $\bar{\mu} \rightarrow 0$. Since the coefficient of z^m is not equal to zero for sufficiently small $\bar{\mu}$, we obtain that equation (13) has m roots with moduli less than unity and n roots with moduli greater than unity.

From the known relation between fixed points of a mapping and periodic solutions we obtain that from Γ_0 there is born a unique periodic motion of saddle type.

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Note: Figure translations are in progress. See original paper for figures.

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