

THE DIRICHLET PROBLEM FOR STRONGLY COUPLED SYSTEMS OF ELLIPTIC TYPE

Nguyen Thya Hop

1966

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Abstract

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UDC 517.946

MATHEMATICS

NGUEN TKHYA KHOSH

THE DIRICHLET PROBLEM FOR STRONGLY COUPLED SYSTEMS OF ELLIPTIC TYPE

(Presented by Academician M. A. Lavrent'ev, 28 I 1966)

Let an elliptic system be given

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0, \quad (1)$$

$$\det[A + 2B\lambda + C\lambda^2] \neq 0$$

for real λ ; $\det C \neq 0$; A, B, C are given constant matrices; u is the unknown vector with components $u_1, u_2, \dots, u_n$.

For a system of the form (1), A. V. Bitsadze ⁽²⁾ introduced the concepts of weak and strong coupling, which proved useful in the study of linear boundary-value problems for the indicated system. As is known, the Dirichlet problem for weakly coupled systems is Fredholm ⁽⁶⁾ (see also ⁽⁴⁾). In ⁽²⁾ examples are given of strongly coupled systems of two equations for which the Dirichlet problem ceases to be Fredholm and Noetherian. In ⁽⁷⁾ the Dirichlet problem for a strongly coupled system in the disk $|z| < 1$ was investigated in the case when i is an n -fold root of the characteristic equation

$$\det[A + 2B\lambda + C\lambda^2] = 0. \quad (2)$$

The present paper is devoted to the study of the Dirichlet problem for strongly coupled systems (1) of two equations in the general case.

Introduce the differential operators

$$L_1(w) = \partial^2 w / \partial \bar{z}^2 + \sigma \partial^2 w / \partial z \partial \bar{z} + \pi \partial^2 w / \partial z^2,$$

$$\bar{L}_1(w) = \partial^2 w / \partial z^2 + \bar{\sigma} \partial^2 w / \partial z \partial \bar{z} + \bar{\pi} \partial^2 w / \partial \bar{z}^2,$$

where $z = x + iy$, $\bar{z} = x - iy$, $w = u_1 + iu_2$.

Theorem 1. In order that system (1), written in complex form

$$\begin{aligned} L(w) = & a \partial^2 w / \partial z^2 + 2b \partial^2 w / \partial z \partial \bar{z} + c \partial^2 w / \partial \bar{z}^2 + \\ & + d \partial^2 \bar{w} / \partial z^2 + 2e \partial^2 \bar{w} / \partial z \partial \bar{z} + f \partial^2 \bar{w} / \partial \bar{z}^2 = 0, \end{aligned} \quad (3)$$

be elliptic and strongly coupled, it is necessary and sufficient that it have the form

$$L(w) = \alpha L_1(w) + \beta L_1(\bar{w}) + \gamma \bar{L}_1(\bar{w}) + \delta \bar{L}_1(w) = 0, \quad (4)$$

where $\alpha, \beta, \gamma, \delta, \sigma, \pi$ are complex numbers satisfying the conditions:

$$\begin{aligned} 1. \quad & \alpha\delta = \beta\gamma, \\ 2. \quad & |\alpha|^2 - |\beta|^2 + |\gamma|^2 - |\delta|^2 \neq 0; \\ 3. \quad & (|\sigma|^2 - |\sigma^2 - 4\pi|)/2 < 1 + |\pi|^2. \end{aligned} \quad (5)$$

Theorem 2. In order that system (3) be elliptic and strongly coupled, it is necessary and sufficient that the conditions hold:

$$\begin{aligned} 1. \quad & (f\bar{f} - d\bar{d})(\bar{b}c - b\bar{a}) - (c\bar{c} - a\bar{a})(\bar{e}f - e\bar{d}) = 0. \\ 2. \quad & [(c\bar{c} + a\bar{a}) - (d\bar{d} + f\bar{f})][(c\bar{c} - a\bar{a})^2 + (d\bar{d} - f\bar{f})^2] \neq 0. \\ 3. \quad & |r + p\bar{r}|^2 + |(r + p\bar{r})^2 - 4p| < 2(1 + |p|^2). \end{aligned}$$

Here p is a root of the quadratic equation

$$p^2(\bar{a}c - d\bar{f}) + p\{(a\bar{a} + c\bar{c}) - (d\bar{d} + f\bar{f})\} + \bar{c}a - \bar{f}d = 0,$$

$$r = \begin{cases} 2 \frac{\bar{b}c - b\bar{a}}{c\bar{c} - a\bar{a}}, & \text{if } c\bar{c} \neq a\bar{a}, \\ 2 \frac{e\bar{f} - \bar{e}d}{f\bar{f} - d\bar{d}}, & \text{if } f\bar{f} \neq d\bar{d}. \end{cases}$$

When condition (5) is satisfied, the strongly coupled elliptic system

$$L_1(W) = 0 \quad (6)$$

will below be called **fundamental**.

In the notation $\sigma = \nu_1 + \nu_2$, $\pi = \nu_1\nu_2$, condition (5) is equivalent to the inequalities $|\nu_1| < 1$, $|\nu_2| < 1$ or $|\nu_1| > 1$, $|\nu_2| > 1$. The roots λ_1 and λ_2 ($\text{Im } \lambda_1 > 0$ and $\text{Im } \lambda_2 > 0$) of the characteristic equation (2) are related to ν_1 and ν_2 by the formulas

$$\lambda_1 = -i \frac{\nu_1 + 1}{\nu_1 - 1}, \quad \lambda_2 = -i \frac{\nu_2 + 1}{\nu_2 - 1}.$$

The systems

$$M(w^*) = \bar{\alpha}L_1(w^*) + \beta L_1(\bar{w}^*) + \gamma \bar{L}_1(\bar{w}^*) + \bar{\delta}L_1(w^*) = 0, \quad (7)$$

$$\bar{L}_1(W^*) = 0 \quad (8)$$

are the systems Lagrange-adjoint (in the variables (x, y)) respectively to the systems (4) and (6).

The general solutions of the fundamental system (6) and its adjoint (8), for $\lambda_1 = \lambda_2$, have respectively the form

$$W = \overline{(z - \nu\bar{z})} \Phi(z - \nu\bar{z}) + \Psi(z - \nu\bar{z}), \quad (9)$$

$$W^* = (z - \nu\bar{z}) \overline{\Phi^*(z - \nu\bar{z})} + \overline{\Psi^*(z - \nu\bar{z})}, \quad (10)$$

and for $\lambda_1 \neq \lambda_2$,

$$W = \Phi(z - \nu_1\bar{z}) + \Psi(z - \nu_2\bar{z}), \quad (11)$$

$$W^* = \overline{\Phi^*(z - \nu_1\bar{z})} + \overline{\Psi^*(z - \nu_2\bar{z})}; \quad (12)$$

here $\Phi, \Psi, \Phi^*, \Psi^*$ are arbitrary holomorphic functions of their arguments.

Theorem 3. There exist complex numbers $\varepsilon_1, \varepsilon_2, \tau_1, \tau_2$, satisfying the conditions $|\varepsilon_1| \neq |\varepsilon_2|$, $|\tau_1| \neq |\tau_2|$, such that

$$w = \varepsilon_1 W + \varepsilon_2 \bar{W}, \quad (13)$$

$$w^* = \tau_1 W^* + \tau_2 \bar{W}^*, \quad (14)$$

where w, w^*, W, W^* are the general solutions of the systems (4), (7), (6), (8), respectively.

Let $\mathcal{D}$ be a simply connected domain bounded by a rectifiable curve Γ ; $\mathcal{D}_\zeta, \mathcal{D}_{\zeta_1}, \mathcal{D}_{\zeta_2}$ are the images of the domain $\mathcal{D}$ under the transformations $\zeta = z - \nu\bar{z}$; $\zeta_1 = z - \nu_1\bar{z}$, $\zeta_2 = z - \nu_2\bar{z}$. Solutions W and W^* of the systems (6) and (8) will be called **regular** if in the expressions (9), (10), (11), (12) the functions $\Phi(\zeta), \Psi(\zeta), \Phi^*(\zeta), \Psi^*(\zeta) \in E_2(\mathcal{D}_\zeta)$; $\Phi(\zeta_1), \partial\Phi^*/\partial\zeta_1 \in E_2(\mathcal{D}_{\zeta_1})$, $\Psi(\zeta_2), \partial\Psi^*/\partial\zeta_2 \in E_2(\mathcal{D}_{\zeta_2})$ (for the definition of the class E_2 , see [8]). In accordance with this, solutions w and w^* of the systems (4) and (7) will be called regular—

are, if the functions W and W^* in (13) and (14) are regular solutions of systems (6) and (8).

Definition. By the Dirichlet problem $D(\nu_1, \nu_2)$ in the case $\nu_1 \neq \nu_2$ (respectively, $D(\nu)$ in the case $\nu_1 = \nu_2 = \nu$) for system (4) we shall mean the problem of finding regular solutions of this system satisfying the condition

$$w|_\Gamma = f(s), \quad (15)$$

where $f(s) \in L_2(\Gamma)$. The homogeneous problem corresponding to (4), (15) will be denoted by $D_0(\nu_1, \nu_2)$ ($D_0(\nu)$).

By the homogeneous problem $D_0^*(\nu_1, \nu_2)$, $\nu_1 \neq \nu_2$ ($D_0^*(\nu)$, $\nu_1 = \nu_2 = \nu$), adjoint to $D(\nu_1, \nu_2)$ (to $D(\nu)$), we shall mean the problem of finding regular solutions of system (7) satisfying the condition

$$w^*|_\Gamma = 0. \quad (16)$$

From Green's formula one obtains the necessary solvability condition for the problem $D(\nu_1, \nu_2)$ ($D(\nu)$) in the form

$$\int_\Gamma [fQ(\bar{w}^*) + \bar{f}Q(w^*)] ds = 0, \quad (17)$$

where w^* are regular solutions of the problem $D_0^*(\nu_1, \nu_2)$, and

$$Q(\bar{w}^*) = \overline{\mu_2 Q_1(\theta_2 \omega^* + \theta_1 \bar{\omega}^*)} + \mu_1 Q_1(\theta_2 \omega^* + \theta_1 \bar{\omega}^*),$$

with $|\mu_1| \neq |\mu_2|$, $|\theta_1| \neq |\theta_2|$ being definite constants depending only on the coefficients of the equations, and

$$Q_1(\cdot) = \nu_1 \nu_2 \frac{d\bar{z}}{ds} \frac{\partial}{\partial \bar{z}} - (\nu_1 + \nu_2) \frac{dz}{ds} \frac{\partial}{\partial z} - \frac{dz}{ds} \frac{\partial}{\partial \bar{z}}.$$

If condition (17) also proves sufficient for the solvability of the problem $D(\nu_1, \nu_2)$ ($D(\nu)$), then we shall call this problem **normally solvable in the sense of Hausdorff** (this term was introduced by A. V. Bitsadze ⁽³⁾).

Theorem 4. A. *The spaces of solutions of the homogeneous problems $D_0(\nu_1, \nu_2)$ ($D_0(\nu)$) of systems (4) and (6) are simultaneously either zero-dimensional or infinite-dimensional.*

B. *From the normal solvability of the problem $D(\nu_1, \nu_2)$ ($D(\nu)$) for system (4) there follows the normal solvability of this problem for the fundamental system (6), and conversely.*

This theorem answers the question posed in ⁽⁵⁾: the homogeneous Dirichlet problem for the equation

$$\frac{\partial}{\partial z} \left(\frac{\partial w}{\partial z} + \lambda \frac{\partial w}{\partial \bar{z}} \right) = 0, \quad 0 < |\lambda| < 1,$$

cannot have a finite number of nonzero linearly independent solutions.

By virtue of Theorem 4, in studying the problem $D(\nu_1, \nu_2)$ ($D(\nu)$) for the strongly coupled system (3), it suffices to restrict ourselves to the study of this problem for the fundamental system (6).

The Dirichlet problem $D(\nu)$ for the fundamental system (6) is the Dirichlet problem for the system

$$w_{z_\lambda \bar{z}_\lambda} = 0, \quad z_\lambda = x + \lambda y,$$

and it was studied in ⁽⁹⁾. Thus, below we shall consider only the problem $D(\nu_1, \nu_2)$, $\nu_1 \neq \nu_2$.

Theorem 5. The homogeneous problem $D_0(\nu_1, \nu_2)$ in the disk has an infinite set of linearly independent solutions if and only if ν_1/ν_2 ($\nu_2 \neq 0$) is equal to a root of unity.

In studying the problem $D(\nu_1, \nu_2)$ in an ellipse, without loss of generality one may consider only the case when the equation of the ellipse has the form

$$mz^2 + \bar{m}\bar{z}^2 + 2z\bar{z} = n^2, \quad |m| < 1, \quad (18)$$

where n is a real number.

Let $p = 1 + \sqrt{1 - |m|^2}$.

Theorem 6. In a finite domain bounded by the ellipse (18), the homogeneous problem $D_0(\nu_1, \nu_2)$ has an infinite set of linearly independent solutions if and only if

$$\frac{\bar{m} + \nu_1 p}{p + \nu_1 m} \frac{p + \nu_2 m}{m + \nu_2 p} \quad (\neq 1)$$

is a root of unity.

Theorem 7. The problem $D(\nu_1, \nu_2)$ is not normally solvable (in the sense of Hausdorff) in a domain with analytic boundary.

Theorem 8. In a finite domain $\widetilde{D}$ bounded by a Lyapunov curve with exponent $> 1/2$, the problem $D(\nu_1, \nu_2)$ is not Noetherian.

The proof of the theorems formulated above is carried out by reducing the problem $D(\nu_1, \nu_2)$ to an equivalent Fredholm integral equation of the first kind.

Novosibirsk State
University

Received
26 I 1966

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