

EXACT UNIQUENESS CLASSES FOR THE SOLUTION OF THE CAUCHY PROBLEM FOR SECOND-ORDER EQUATIONS

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Abstract

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MATHEMATICS

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EXACT UNIQUENESS CLASSES FOR THE SOLUTION OF THE CAUCHY PROBLEM FOR SECOND-ORDER EQUATIONS

(Presented by Academician I. G. Petrovskii on 21 II 1966)

In 1936, in the paper ⁽¹⁾, a necessary and sufficient condition was established for uniqueness of the solution of the Cauchy problem for the parabolic equations $\partial u / \partial t = (-1)^{p-1} \partial^{2p} u / \partial x^{2p}$ in the class of functions which, for each $t \geq 0$ and $|x| \rightarrow \infty$, grow no faster than $\exp\{k|x|h(|x|)\}$, where $h(x) > 0$ is a nonincreasing function, $k > 0$. This condition turned out to be the divergence of the integral

$$\int_{-\infty}^{\infty} [h(x)]^{1-2p} dx.$$

As was subsequently clarified ⁽²⁾, for equations with constant coefficients the uniqueness classes for the solution of the Cauchy problem do not depend on the type of the equation. For such equations, necessary and sufficient conditions for uniqueness of the solution of the Cauchy problem, generalizing the result formulated above, were obtained in the papers ^(3, 4). In addition, in the papers of a number of authors ⁽⁵⁻⁷⁾, sufficient conditions for uniqueness of the solution of the Cauchy problem were established for parabolic equations in the sense of I. G. Petrovskii with coefficients depending on x (bounded, or growing as $|x| \rightarrow \infty$ no faster than some fixed power of $|x|$).

In the present paper we consider an equation of the form

$$\partial u(x, t) / \partial t = a \partial^2 u(x, t) / \partial x^2 + q(x) u(x, t), \quad (1)$$

$$-\infty < x < \infty, \quad 0 \leq t < \infty, \quad a \neq 0,$$

with the initial condition

$$u(x, 0) = u_0(x). \quad (2)$$

Without making any assumptions about the type of equation (1) (a is any complex number) and without restricting the growth of $|q(x)|$ as $|x| \rightarrow \infty$, we establish necessary and sufficient conditions for uniqueness of solutions of normal type * (8) of problem (1)–(2) in various classes of functions, depending on the behavior of $|q(x)|$ as $|x| \rightarrow \infty$.

It turns out that if $|q(x)|$ does not grow too rapidly as $|x| \rightarrow \infty$, then the uniqueness classes for the solution of problem (1)–(2) depend neither on the number a , nor on the function $q(x)$; namely, the following holds:

Theorem 1. *Let the function $q(x)$ satisfy the conditions:*

- 1) $q(x)$ is a real, twice continuously differentiable function;
- 2) the functions $q'^2|q|^{-5/2}$ and $q''|q|^{-3/2}$ belong to $L(-\infty, -x_0)$ and $L(x_0, \infty)$ for sufficiently large x_0 ;
- 3) either $q(x) \leq q_1 = \text{const}$ for $-\infty < x < \infty$, or $q(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$;
- 4)

$$\int_{-\infty}^{\infty} |q(x)|^{-1/2} dx = \infty, \quad \int_{-\infty}^{\infty} |q(x)|^{-1/2} dx = \infty;$$

* In what follows only such solutions are considered.

- 5) $\sqrt{|q(x)|} = o(H(|x|))$ as $|x| \rightarrow \infty$ for every monotone function $H(x) > 0$ satisfying the conditions $\int_0^\infty \frac{dx}{H(x)} < \infty$, $|x|H(x) \leq C \left| \int_0^x H(t) dt \right|$.

In the cases $\arg a = 0$, $q(x) \rightarrow +\infty$ and $\arg a = \pi$, $q(x) \leq q_1$, the function $q(x)$ must also satisfy the following condition:

- 6) $q'(x)$ preserves its sign for sufficiently large values of $|x|$, or the weaker condition
- 6') there exists a function $h_1(x)$, monotone for $x > 0$, satisfying the condition $\int_0^\infty [h_1(x)]^{-1} dx = \infty$, and such that for all x , $-\infty < x < \infty$,

$$\sqrt{|q(x)|} \leq A_1 h_1(|x|) + A_2, \quad A_1, A_2 > 0.$$

Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions

$$|f(x)| \leq C_f \exp \left| \int_0^x h(t) dt \right|, \quad (3)$$

where $h(x) > 0$ is a monotone, even function, it is necessary and sufficient that the integral $\int_0^\infty [h(x)]^{-1} dx$ diverge.

We note that the necessity of this condition can be proved only under assumptions 1)–5), whereas in establishing its sufficiency one can substantially weaken condition 5), replacing it by the assumption $|q(x)| \leq A_3 \exp\{A_4 Q(|x|)\}$, $A_3, A_4 > 0$, where $Q(x) > 0$ is nondecreasing and

$$\int_{-\infty}^{\infty} x^{-3} Q(x) dx < \infty.$$

For more rapid growth of $|q(x)|$ as $|x| \rightarrow \infty$, when condition 4) is not fulfilled, the uniqueness classes of the solution of the Cauchy problem (1)–(2) depend both on $|q(x)|$ and on the coefficient a , and, moreover, on the summability as $|x| \rightarrow \infty$ of the function $|q|^{-1/2} \ln |q|$.

Theorem 2. Let the function $q(x)$ satisfy conditions 1) and 2) of Theorem 1 and, moreover, the conditions:

- 3) $q(x) \rightarrow -\infty$ or $q(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$;
- 4) $q'(x)$ preserves its sign for sufficiently large values of $|x|$;
- 5) $\int_{-\infty}^{\infty} |q(x)|^{-1/2} \ln |q(x)| dx < \infty$ and $\int_{-\infty}^{\infty} |q(x)|^{-1/2} \ln |q(x)| dx < \infty$.

Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions

$$|f(x)| \leq C_f \frac{\alpha(x)}{\sqrt[4]{1 + |q(x)|}} \exp \left\{ \mu \left| \int_0^x \sqrt{|q(t)|} dt \right| \right\}, \quad (4)$$

where

$$\mu = \begin{cases} |a|^{-1/2} \sin \frac{1}{2} |\arg a|, & \text{when } q(x) \rightarrow \infty, \\ |a|^{-1/2} \cos \frac{1}{2} \arg a, & \text{when } q(x) \rightarrow -\infty, \end{cases}$$

and $\alpha(x) > 0$ is an even, continuous function, it is necessary and sufficient that $\inf \alpha(x) = 0$.

We note that in the case $\arg a = \pi$, $q(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$, the theorem remains true even if condition 5) is replaced by the weaker condition

$$\int_{-\infty}^{\infty} |q(x)|^{-1/2} dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |q(x)|^{-1/2} dx < \infty.$$

We also note that the proof of sufficiency of the condition $\inf \alpha(x) = 0$ does not use assumptions 4) and 5) of the theorem.

Theorem 3. Let the function $q(x)$ satisfy conditions 1)–4) of Theorem 2 and, in addition, the conditions:

- 5) $\int_{-\infty}^{\infty} |q(x)|^{-1/2} dx < \infty, \quad \int_{-\infty}^{\infty} |q(x)|^{-1/2} \ln |q(x)| dx = \infty, \quad \int_{-\infty}^{\infty} |q(x)|^{-1/2} dx < \infty,$

$$\int_{-\infty}^{\infty} |q(x)|^{-1/2} \ln |q(x)| dx = \infty;$$

6) $\arg a \neq 0$ when $q(x) \rightarrow -\infty$, and $\arg a \neq \pi$ when $q(x) \rightarrow +\infty$.

Then, for uniqueness of the solution of the Cauchy problem (1)–(2) in the class of functions

$$|f(x)| \leq C_f \exp \left| \int_0^x [\mu \sqrt{|q(t)|} + h(t)] dt \right|, \quad (5)$$

where $h(x) > 0$ is an even, monotone function, it is necessary and sufficient that

$$\int_{-\infty}^{\infty} |q(x)|^{-1/2} \ln \frac{\sqrt{|q(x)|}}{h(x)} dx = +\infty, \quad \int_{-\infty}^{\infty} |q(x)|^{-1/2} \ln \frac{\sqrt{|q(x)|}}{h(x)} dx = +\infty. \quad (6)$$

We note that the proof of the necessity of this condition also uses the “identical” behavior of $|q(x)|$ as $x \rightarrow +\infty$ and $x \rightarrow -\infty$, in the sense that, for any even monotone function $h(x)$, failure of either one of the two conditions (6) entails the failure of the other. We note, moreover, that the necessity of condition (6) also holds in the case $\arg a = 0$, $q(x) \rightarrow -\infty$.

Theorems 2 and 3 show that the growth of $|q(x)|$ greater than that allowed by condition 4) of Theorem 1 leads to an enlargement of the uniqueness classes (4) and (5) in comparison with the class (3) (except in the cases when $\mu = 0$). This “dissipative” effect has a transparent physical meaning for the heat equation $\partial u / \partial t = \partial^2 u / \partial x^2 + q(x)u$, for which the condition $q(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$ means increasing heat loss (dissipation of heat). It is clear that the uniqueness classes for the solution of the Cauchy problem must then become larger. In the case of the heat equation, the corresponding uniqueness theorem was obtained by the author in (9).

In the cases when $\mu = 0$, i.e. when equation (1) is the heat equation and $q(x) \rightarrow +\infty$, or when (1) is the backward heat equation and $q(x) \rightarrow -\infty$, Theorems 2 and 3 show that the uniqueness classes (4) and (5) are narrower than the class (3) and become narrower as the growth of $|q(x)|$ increases.

Thus, for the heat equation, if $q(x) \rightarrow +\infty$ sufficiently rapidly (the conditions of Theorem 2 are satisfied), the Cauchy problem has a unique solution in the class of functions

$$|f(x)| \leq C_f \alpha(x) [1 + |q(x)|]^{-1/4}$$

if and only if $\inf \alpha(x) = 0$. Hence, for example, it follows that in this case in the class $L^2(-\infty, \infty)$ uniqueness of the solution of the Cauchy problem does not hold.

The proofs of Theorems 1, 2, 3 proceed according to a common scheme. The Cauchy problem (1)–(2) is considered in a certain Banach space; this space is the collection of all functions $f(x)$, continuous for $-\infty < x < \infty$, with norm

$$\|f\| = \sup_x [|f(x)|g(x)];$$

in the case of Theorem 1

$$g(x) = \exp \left\{ - \left| \int_0^x h(t) dt \right| \right\},$$

in the case of Theorem 2

$$g(x) = \frac{\sqrt[4]{1 + |q(x)|}}{\alpha(x)} \exp \left\{ -\mu \left| \int_0^x \sqrt{|q(t)|} dt \right| \right\},$$

and in the case of Theorem 3

$$g(x) = \exp \left\{ - \left| \int_0^x [\mu \sqrt{|q(t)|} + h(t)] dt \right| \right\}.$$

By virtue of the well-known theorem of Hille ⁸, the question of uniqueness of the normal solution of problem (1)–(2) in a Banach space reduces to the study of the behavior of $\|y(x, \lambda)\|$ (as $\operatorname{Re} \lambda \rightarrow +\infty$), where $y(x, \lambda)$ is a solution of the equation

$$ay'' + q(x)y = \lambda y. \quad (7)$$

This study is carried out on the basis of the asymptotics, obtained by us, of the solutions of equation (7) as $\operatorname{Re} \lambda \rightarrow +\infty$, which is described by the following theorem.

Theorem 4. *Let the function $q(x)$ satisfy conditions 1), 2), and 3) of Theorem 1. Then equation (7) has solutions ($j = 1, 2$)*

$$y_j(x, \lambda) = \frac{1}{\sqrt{W_1(x, \lambda)}} (1 + \varepsilon_j(x, \lambda)) \exp \int_{x_0}^x W_j(t, \lambda) dt, \quad (8)$$

where

$$y'_j(x, \lambda) = \sqrt{W_1(x, \lambda)} [(-1)^{j-1} + \varepsilon_{j+2}(x, \lambda)] \exp \int_{x_0}^x W_j(t, \lambda) dt, \quad (9)$$

where $W_j(x, \lambda) = (-1)^{j-1} \sqrt{\lambda - q(x)}$ ($\operatorname{Re} W_1(x, \lambda) \geq 0$), and the functions $\varepsilon_j(x, \lambda)$ ($1 \leq j \leq 4$) satisfy the conditions:

- 1) if $q(x) \leq q_1$, $-\infty < x < \infty$, then for any $\varepsilon > 0$ one can find such $\sigma_1(\varepsilon)$ that for $\operatorname{Re} \lambda > \sigma_1(\varepsilon)$

$$|\varepsilon_j(x, \lambda)| < \varepsilon, \quad j = 1, 2, 3, 4. \quad (10)$$

- 2) if $q(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$, then for any $\varepsilon > 0$ and $\gamma > 0$ one can specify $\sigma_1(\varepsilon, \gamma)$ and $\sigma_2(\varepsilon)$ such that (10) holds in the regions: a) $-\infty < x < \infty$, $\operatorname{Re} \lambda \geq \sigma_1(\varepsilon, \gamma)$, $|\arg \lambda| \geq \gamma$; b) $x \geq p(2\sigma)$, $\operatorname{Re} \lambda \geq \sigma_2(\varepsilon)$, $|\arg \lambda| \leq \gamma$; c) $x \leq \tilde{p}(2\sigma)$, $\operatorname{Re} \lambda \geq \sigma_2(\varepsilon)$, $|\arg \lambda| \leq \gamma$. Here $p(y)$ and $\tilde{p}(y)$ are determined from the conditions $q(x) \geq y$ for $x \geq p(y) > 0$ and for $x \leq \tilde{p}(y) < 0$.

In formulas (8) and (9), in cases 1) and 2a) one should put $x_0 = 0$, in case 2b) $x_0 = p(2\sigma)$, and in case 2c) $x_0 = \tilde{p}(2\sigma)$.

Let us note that the asymptotics of the solutions of equation (7), analogous to (8), (9), for fixed λ and $x \rightarrow \infty$, is well known^{10,11}.

Theorem 4 makes it possible to decide the question of the existence of a non-trivial solution $y(x, \lambda)$ (of equation (7)), holomorphic in some right half-plane and satisfying there, for some $C > 0$, the condition $\|y(x, \lambda)\| < C$, which also leads to the results of Theorems 1, 2, and 3.

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Note: Figure translations are in progress. See original paper for figures.

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