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Abstract

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MATHEMATICS

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UNIFORM APPROXIMATIONS BY RATIONAL FUNCTIONS (ALGEBRAIC AND TRIGONOMETRIC) AND GLOBAL FUNCTIONAL PROPERTIES

(Presented by Academician A. N. Kolmogorov on 18 V 1965)

Without dwelling on the history of the question (it may be found in (5)), let us recall some definitions and notation.

Let $\Phi(u)$ be increasing, continuous, and convex downward on $[0, \infty)$, $\Phi(0) = 0$. We say that Φ has the Δ_2 -property, $\Phi \in \Delta_2$, if for all $u \geq 0$ we have $\Phi(2u) \leq C\Phi(u)$, where $C = C(\Phi) = \text{const}$. The Φ -variation of a function $f(x)$ ($x \in [a, b]$, $-\infty \leq a < b \leq +\infty$) is defined as follows (see (1)):

$$V_\Phi(f) = V_\Phi(f, [a, b]) = \sup \left\{ \sum_{k=1}^n \Phi(|f(x_k) - f(x_{k-1})|) \right\}, \quad (1)$$

where the least upper bound is taken over all $\{x_k\} : a = x_0 < x_1 < \dots < x_n = b$, $n = 1, 2, \dots$

If $V_\Phi(f) < \infty$, then $f \in V_\Phi$. If $kf \in V_\Phi$ for some $k = \text{const} > 0$, then $f \in V_\Phi^*$. Obviously, $V_\Phi \subset V_\Phi^*$ and $V_\Phi = V_\Phi^*$ for $\Phi \in \Delta_2$. Note that any function f continuous on $[a, b]$ belongs to some V_Φ^* . If, following Musielak and Orlicz (1), one introduces the norm

$$\|f\|_\Phi = \|f\|_{\Phi, [a, b]} = \inf\{k : k > 0, V_\Phi(f/k) \leq 1\} \quad (2)$$

and regards $f_1 = f_2$ when $f_1(x) - f_2(x) \equiv \text{const}$, then V_Φ^* becomes a complete linear normed space (with the usual addition of functions and multiplication of a function by a number) ((1), Sec. 3.21), and, if $\|f - f_n\|_\Phi \rightarrow 0$, then $V_\Phi(k(f - f_n)) \rightarrow 0$ for some $k = \text{const} > 0$ ((1), Sec. 3.11).

A function $f(x)$ ($x \in [a, b]$) is called Φ -absolutely continuous, $f \in AC_\Phi$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum \Phi(|f(\beta_i) - f(\alpha_i)|) < \varepsilon$ whenever $\sum \Phi(\beta_i - \alpha_i) < \delta$, $a \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots \leq b$. $f \in AC_\Phi^*$ if $kf \in AC_\Phi$

for some $k = \text{const} > 0$. Obviously, $AC_\Phi \subset V_\Phi$, $AC_\Phi^* \subset V_\Phi^*$, $AC_\Phi \subset AC_\Phi^*$, and $AC_\Phi = AC_\Phi^*$ for $\Phi \in \Delta_2$. AC_Φ^* is a closed subspace of the space V_Φ^* ((1), Sec. 3.21).

We say that $f \in L_\Phi[a, b] = L_\Phi$ if the function $\Phi(|f(x)|)$ is summable on $[a, b]$, and $f \in L_\Phi^*[a, b] = L_\Phi^*$ if $kf \in L_\Phi$ for some $k = \text{const} > 0$. The norm in L_Φ^* is defined as follows (see (3)):

$$\|f\|_{L_\Phi[a, b]} = \inf \left\{ k : k > 0, \int \Phi(|f|/k) dx \leq 1 \right\}. \quad (3)$$

L_Φ^* (the Orlicz space) is a complete normed linear space. If $\Phi \in \Delta_2$, then $L_\Phi = L_\Phi^*$. If $\Phi(u) = u^p$, $p \geq 1$, then $L_\Phi = L^p$, and $\|f\|$ in the metric L_Φ^* coincides with $\|f\|$ in L^p .

* For example, with $\Phi(u) = \tilde{\omega}^{-1}(u/3)$, where $\tilde{\omega}$ is a regularization of the modulus of continuity of the function f ; see (4), Sec. 3.3.

For $\delta > 0$ (and $f_h(x) = f(x+h)$) the quantity

$$\omega_\Phi(\delta, f) = \omega_\Phi(\delta, f[a, b]) = \sup_{|h| \leq \delta} \|f - f_h\|_{L_\Phi([a, b] \cap [a-h, b-h])} \quad (4)$$

will be called the Φ -modulus of continuity of the function f (on $[a, b]$). $R_n = R_n[f] = R_n[f, [a, b]]$ denotes the least deviation (in the uniform metric) of the function f from real rational functions (algebraic or trigonometric) of degree (order) not exceeding n^* (see (5)); $r_n(x) = r_n(x, f)$ is a rational function of order $\leq n$ for which $\max |f(x) - r_n(x)| = R_n$. In the case of rational trigonometric functions we always assume $0 \leq a < b \leq 2\pi$.

1°. Rate of approximation and Φ -variation.

Lemma 1. For every rational (algebraic or trigonometric) function $r(x)$ we have:

$$V_\Phi(r) \leq 2n\Phi(2M) \quad (5a)$$

$$\|r\|_\Phi \leq 4M/\Phi^{-1}(1/n), \quad (5b)$$

where n is the order of the function r , $M = \max\{|r(x)| : x \in [a, b]\}$.

Proof. From the increase of Φ it follows that the sum (1) will not decrease if to the partition points x_k one adds a point y , $x_{k-1} < y < x_k$, at which $r(y) \in [r(x_{k-1}), r(x_k)]$.

From the convexity downward of Φ it follows that this sum will not decrease if one removes a partition point x_k for which $r(x_k) \in [r(x_{k-1}), r(x_{k+1})]$.

Thus,

$$V_{\Phi}(r) = \sum \Phi(|r(y_i) - r(y_{i-1})|),$$

where the y_i are some of the points $y \in [a, b]$ at which r attains local extrema ($y_0 = a < y_1 < \dots < y_k = b$, $k \leq 2n$). Hence we obtain (5a). From this and (2) we have: $\|r\|_{\Phi} \leq 2M/\Phi^{-1}(1/2n)$. Taking into account that Φ^{-1} is convex upward and increasing, we obtain (5b).

Remark. The preceding arguments show that if the graph of the function $y = f(x)$ can be divided into n intervals of monotonicity, then estimate (5) remains valid ($r = f$). If, in addition, f is continuous, then the right-hand sides in inequalities (5) should be halved.

Theorem 1. Always

$$\|f\|_{\Phi} \leq 100 \sum_{n=0}^{\infty} \lambda_n R_n, \quad \text{where} \quad \lambda_{n-1} = \left[n \Phi^{-1} \left(\frac{1}{n} \right) \right]^{-1} \leq [\Phi^{-1}(1)]^{-1}. \quad (6)$$

Proof. Obviously,

$$\|f\|_{\Phi} \leq \|r_1 - r_0\|_{\Phi} + \sum_{k=1}^{\infty} \|r_{2^k} - r_{2^{k-1}}\|_{\Phi}.$$

By Lemma 1,

$$\|r_1 - r_0\|_{\Phi} \leq 8R_0/\Phi^{-1}(1);$$

$$\begin{aligned} \|r_{2^k} - r_{2^{k-1}}\|_{\Phi} &\leq 4 \cdot 2R_{2^{k-1}}/\Phi^{-1}(1/3 \cdot 2^{k-1}) \leq \\ &\leq 48R_{2^{k-1}}/\Phi^{-1}(2^{-(k-2)}) \leq 96 \sum_{2^{k-2}+1}^{2^{k-1}} R_{n-1} \left[n \Phi^{-1} \left(\frac{1}{n} \right) \right]^{-1}. \end{aligned} \quad (7)$$

Theorem 2. If

$$\sum R_n[f] [n \Phi^{-1}(1/n)]^{-1} < \infty,$$

then $f \in AC_{\Phi}^*$. If, moreover, $\rho_n(x)$ are rational functions of order respectively not exceeding n , for which (with some $C = \text{const}$)

$$\max\{|f(x) - \rho_n(x)| : x \in [a, b]\} \leq CR_n[f],$$

then $\|f - \rho_n\|_\Phi \rightarrow 0$ and $V_\Phi(k(f - \rho_n)) \rightarrow 0$ for some $k = \text{const} > 0$ ($n \rightarrow \infty$).

* $r(x) = P(x)/Q(x)$ is a rational algebraic (trigonometric) function of order $\leq n$, if P and Q are algebraic (respectively trigonometric) polynomials of order $\leq n$.

Proof. Choose natural p from the condition $2^{p-1} \leq n < 2^p$. From the inequality

$$\|f - \rho_n\|_\Phi \leq \sum_{p+1}^{\infty} \|\rho_{2^q} - \rho_{2^{q-1}}\|_\Phi + \|\rho_{2^p} - \rho_n\|_\Phi$$

and estimates of type (7), we find that $\|f - \rho_n\|_\Phi \rightarrow 0$, whence $f \in AC_\Phi^*$ (obviously, $\rho_n \in AC_\Phi$).

The case $\Phi(r) = r$ was considered by the author in (5). In this case the condition of the theorem cannot be weakened. The constructions preceding Theorem 2 of paper (5) show that, in the general case, if $a_0 \geq a_1 \geq \dots > 0$ and $\sum \Phi(a_n) = \infty$, there always exists a function f , not belonging to V_Φ^* , for which $R_n[f] \leq a_n$.

Thus, if $R_n[f] \leq Cn^{-1/p-\varepsilon}$ ($p \geq 1$; $C, \varepsilon > 0$), then $f \in AC_\Phi$ with $\Phi(u) = u^p$.^{*} However, the condition $R_n[f] \leq Cn^{-1/p}$ is insufficient even for f to belong to V_Φ^* .

2°. Rate of approximation and the integral modulus of continuity.

Lemma 2a.^{**} For any function $f(x)$ ($-\infty \leq a \leq x \leq b \leq +\infty$), for $0 < \delta < b - a$ we have:^{**}

$$1) \quad \int_a^{b-\delta} \Phi(|f(x+\delta) - f(x)|) dx \leq 2V_\Phi(f)\delta; \quad (8)$$

$$2) \quad \omega_\Phi(\delta, f) \leq \Omega/\Phi^{-1}((b-a)^{-1}), \quad (9)$$

where Ω is the total oscillation of f on $[a, b]$; $\Omega = \max f - \min f$.

Proof. If $a, b \neq \pm\infty$, put $x_k = a + k\delta$ for $k = 0, 1, \dots, n-1 = [(b-a)/\delta]$ and $x_n = b - \delta$. Let $\sum'$ and $\sum''$ denote, respectively, summation over odd and over even k . Then

$$\int_a^{b-\delta} \Phi(|f(x+\delta) - f(x)|) dx = \sum \int_{x_{k-1}}^{x_k} \leq$$

$$\leq \left(\sum' + \sum'' \right) \sup_{x_{k-1} \leq x \leq x_k} \Phi(|f(x+\delta) - f(x)|) \delta \leq 2V_\Phi(f)\delta,$$

and (8) is proved. The cases $a = -\infty$, $b = \infty$ are obtained by passage to the limit. Since always the integral in (8) is $\leq \Phi(\Omega)(b-a)$, (9) follows from (4) and (3). As a consequence of (8) and (5a) we obtain

Lemma 2. Let $r(x)$ be a rational (algebraic or trigonometric) function of order $\leq n$, whose modulus on $[a, b]$ does not exceed M ($-\infty \leq a < b \leq +\infty$). Then

$$\int_a^{b-\delta} \Phi(|r(x+\delta) - r(x)|) dx \leq 4n\Phi(2M)\delta, \quad (10)$$

$$\omega_\Phi(\delta, r) \leq 8M/\Phi^{-1}(1/n\delta). \quad (11)$$

Remark. These estimates are, obviously, also valid for any functions $r = f$ (not necessarily continuous) of the kind discussed in the remark to Lemma 1.

Theorem 3. For any function $f(x)$ ($-\infty < a \leq x \leq b < \infty$), the inequality holds

$$\omega_\Phi(\delta, f) \leq C \sum_{0 < n < \delta^{-1}} \lambda_n(\delta) R_n, \quad \lambda_{n-1}(\delta) = [n\Phi^{-1}(1/n\delta)]^{-1}, \quad (12)$$

where $C = C(\Phi, b-a)$.

* It is easy to see that in this case $\|f\|_\Phi = \sup(\sum |f(x_i) - f(x_{i-1})|^p)^{1/p}$.

** This lemma generalizes Lemma 2 of B. I. Golubov's paper (?), in which $\Phi(u) = u^p$, $p \geq 1$. Still earlier the case $\Phi(u) = u$ was noted by P. L. Ul'yanov.

Proof. Let $2^{-k-1} < \delta \leq 2^{-k}$. Obviously,

$$\omega_\Phi(\delta, f) \leq \omega_\Phi(\delta, f - r_{2^k}) + \sum_{p=1}^k \omega_\Phi(\delta, r_{2^p} - r_{2^{p-1}}) + \omega_\Phi(\delta, r_1 - r_0).$$

From (9) and (11) we have

$$\omega_\Phi(\delta, f - r_{2^k}) \leq \frac{2R_{2^k}}{\Phi^{-1}((b-a)^{-1})} \leq \frac{4\Phi^{-1}(4)}{\Phi^{-1}((b-a)^{-1})} \cdot 2^{k-1} \frac{R_{2^k}}{2^k \Phi^{-1}((\delta 2^{k-1})^{-1})},$$

$$\omega_{\Phi}(\delta, r_{2^p} - r_{2^{p-1}}) \leq \frac{16R_{2^{p-1}}}{\Phi^{-1}((6 \cdot 2^{p-2}\delta)^{-1})} \leq 192 \frac{2^{p-2}R_{2^{p-1}}}{2^{p-1}\Phi^{-1}((2^{p-2}\delta)^{-1})}.$$

Hence we obtain (12).

Corollary. Let $\omega^{(1)}(\delta, f)$ be the modulus of continuity of the function $f(x)$ ($x \in [a, b]$) in the metric L ($L = L_{\Phi}$ for $\Phi(r) = r$). Then

$$\omega^{(1)}\left(\frac{1}{n}, f\right) \leq C \cdot \frac{1}{n}(R_0[f] + R_1[f] + \dots + R_{n-1}[f]),$$

where $C = 200(1 + b - a)$. In particular, if $R_n[f] = O(n^{-\alpha})$ ($0 < \alpha < 1$), then f satisfies on $[a, b]$ the Lipschitz-Hölder integral condition of order α : $f \in \text{Lip}(\alpha, 1)$.

This theorem is close in form to the theorems of S. N. Bernstein, A. F. Timan, and S. B. Stechkin (see ⁽⁶⁾, p. 344) for approximations by trigonometric polynomials.

Remarks.

1. The results of this paper remain valid when the interval $[a, b]$ is replaced by an arbitrary measurable set $E \subset (-\infty, \infty)$ (cf. ⁽⁵⁾), (if instead of the boundedness of $[a, b]$ one requires the boundedness of E).
2. Theorems 1-3 remain valid if by r_n, ρ_n one understands functions whose graphs are polygonal lines with the number of links $\leq n$, or if one understands polynomials in a Haar, Walsh, etc., system (see the remarks to Lemmas 1, 2), and by R_n the corresponding deviations from f .
3. Let $E_n[f]$ be the least deviation of f from polynomials (algebraic or trigonometric) of degree $\leq n$, and let Φ be as at the beginning of this paper. Then for every Φ there exists $f(x)$ ($x \in [0, 1]$) with the following properties:

- a) $R_n[f] \leq E_n[f] \leq CR_n[f]$ ($C = \text{const}$, $n \geq 0$);
- b) $f \in V_{\Phi}^*$, but $f \notin V_{\Psi}^*$, if $\Psi(u)$ grows near zero essentially more slowly than $\Phi(u)$ (i.e., $\Psi(u)/\Phi(ku) \rightarrow \infty$ for every fixed $k > 0$ as $u \rightarrow 0$; this guarantees the strict inclusion $V_{\Psi}^* \subset V_{\Phi}^*$).

Thus the classes V_{Φ}^* capture something characteristic precisely of rational (and not polynomial) approximations. By substituting $\omega(\delta) = \Phi^{-1}(\delta)$, this assertion follows from the following fundamental result:

Theorem 4. For every modulus of continuity $\omega(\delta)^*$ there exists a function $f(x)$ with the properties:

- a) its ordinary modulus of continuity $\omega(\delta, f)$ has order $\omega(\delta)$:

$$C_1\omega(\delta) \leq \omega(\delta, f) \leq C_2\omega(\delta);$$

b)

$$R_n[f] \leq E_n[f] \leq CR_n[f], \quad R_{2,9^k}[f] = E_{2,9^k}[f]$$

(C, C_1, C_2 are constants > 0 ; $n, k = 0, 1, \dots$).

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* $\omega(\delta)$ is a modulus of continuity if $0 = \omega(+0) \leq \omega(\delta) \leq \omega(\delta+\eta) \leq \omega(\delta)+\omega(\eta)$ for all $\delta, \eta \geq 0$.

Note: Figure translations are in progress. See original paper for figures.

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