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Abstract

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MATHEMATICS

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POSITIVE MIXED PROBLEMS FOR CERTAIN HYPERBOLIC SYSTEMS

(Presented by Academician I. G. Petrovskii, 28 X 1965)

In this note a mixed problem is considered in a cylindrical domain for a general first-order hyperbolic system, assumed to be symmetric (more precisely, Hermitian) only at the points of the lateral surface of the cylinder. An energy inequality and the existence of a strong solution are established under positivity assumptions on the boundary condition of the same kind as in ⁽¹⁻³⁾.

Let G be a bounded domain in the real n -dimensional space R_x^n of points $x = (x^1, \dots, x^n)$, having an $(n-1)$ -dimensional boundary Γ of smoothness class C^2 . In the space $R_{t,x}^{n+1}$ of points (t, x) consider the cylindrical domain $\Omega = I \times G$, where I is the segment $\{t : 0 \leq t \leq T\}$. Denote by Ω' the lateral part $I \times \Gamma$ of the boundary of the domain Ω , by $G(\tau)$ the intersection of the domain Ω with the plane $t = \tau$, and by $\Gamma(\tau)$ the boundary of this intersection. By S denote the unit sphere $|\xi| = 1$ in the space R_ξ^n of points $\xi = (\xi_1, \dots, \xi_n)$.

Suppose that in the closure $\bar{\Omega}$ of the domain Ω there is given a system of first-order partial differential equations

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{\nu=1}^n A_\nu(t, x) \frac{\partial u}{\partial x^\nu} + C(t, x)u = f. \quad (1)$$

Here $u = u(t, x)$ and $f = f(t, x)$ are column vectors of height N with complex components; A_ν and C are square complex matrices of order N . We shall assume that $A_\nu \in C^2(\bar{\Omega})$.* It is sufficient to assume that the elements of the matrix C are bounded measurable functions.

We subject the solution $u(t, x)$ to the boundary condition

$$B(t, x)u(t, x) = g(t, x) \quad \text{on } \Omega', \quad (2)$$

where B is a smooth complex matrix on Ω' , having N columns and a fixed number q of linearly independent rows, and g is a complex column vector of height q , given on Ω' , and also to the initial condition

$$u(0, x) = h(x). \quad (3)$$

We assume system (1) to be hyperbolic in the generalized sense of I. G. Petrovskii⁽⁵⁾ in Ω and Hermitian on Ω' . More precisely, let

$$a(t, x, \xi) = \sum_{\nu=1}^n \xi_{\nu} A_{\nu}(t, x). \quad (4)$$

We shall assume the following to hold.

Condition I. For each point $(t_0, x_0) \in \bar{\Omega}$ there exists a neighborhood U_0 in $R_{t,x}^{n+1}$, and for each point $\xi_0 \in S$ there exists a neighborhood S_0 on S , such that in

$$V_0 = (U_0 \cap \bar{\Omega}) \times S_0$$

there is defined a nonsingular matrix

* Our smoothness assumptions can be weakened.

$r(t, x, \xi)$ of order N and smoothness class $C_{t,x;\xi}^{2;\infty}(V_0)^*$, possessing the following two properties: 1) the matrix rar^{-1} is diagonal and real in V_0 (it suffices to assume it Hermitian); 2) the matrix r is unitary for $(t, x) \in \Omega'$ (so that the A_{ν} are Hermitian on Ω').

Let us note two important cases in which Condition I is satisfied.

1°. The matrices A_{ν} are Hermitian everywhere in $\bar{\Omega}$, so that for r one may take the identity matrix. In this case the theorems formulated below are well known^(1-3,6,7).

2°. The system (1) is strictly hyperbolic in the sense of I. G. Petrovskii⁽⁵⁾ (i.e., the eigenvalues of the matrix (4) are real and pairwise distinct for $(t, x) \in \bar{\Omega}$, $\xi \neq 0$), and, moreover, the matrices A_{ν} are Hermitian on Ω' . In this case, for r^{-1} one must take the matrix whose columns are the normalized linearly independent eigenvectors of the matrix a . If, in addition, $n > 2$ and the domain G is simply connected, then r is immediately constructed globally⁽⁵⁾.

Put $a_l(t, x) = a(t, x, l)$, where $l = (l_1, \dots, l_n)$ is the unit vector of the inner normal to $\Gamma(t)$ at the point $(t, x) \in \Omega'$.

Condition II. $\det a_l(t, x) \neq 0$ on Ω' .

Examples show that there exist strictly hyperbolic systems with infinitely smooth $A_{\nu}(t, x)$, satisfying Conditions I and II and such that, for no nonsingular matrix $T(t, x)$, will the matrices $TA_{\nu}T^{-1}$ ($\nu = 1, \dots, n$) be simultaneously Hermitian in Ω .

For two vectors $u = \{u_j\}$, $v = \{v_j\}$ from C^N put $u \cdot v = \sum u_j \bar{v}_j$. Denote by $\mathfrak{B}(t, x)$ the kernel of the matrix $B(t, x)$ in C^N . Following (1–3), we formulate two more conditions.

Condition III. For every point $(t, x) \in \Omega'$ the quadratic form $u \cdot a_l(t, x)u$ is positive on $\mathfrak{B}(t, x)$:

$$u \cdot a_l(t, x)u \geq p_0 u \cdot u \quad \text{for} \quad 0 \neq u \in \mathfrak{B}(t, x), \quad (5)$$

where $p_0 = \text{const} > 0$.

Condition IV. Condition III is fulfilled, and, moreover, in any $(N - q + 1)$ -dimensional subspace of C^N containing $\mathfrak{B}(t, x)$, there exists a vector v for which $v \cdot a_l(t, x)v < 0$.

Put

$$\|u\|_{G(t)}^2 = \int_{G(t)} |u(t, x)|^2 dx, \quad \|u\|_{\Gamma(t)}^2 = \int_{\Gamma(t)} |u(t, x)|^2 dx',$$

where dx' is the surface element on $\Gamma(t)$. We define $\|g\|_{\Gamma(t)}$ and $\|h\|_G$ analogously.

Theorem 1. *Let Conditions I, II, and III be fulfilled. Then, for solutions $u(t, x) \in C^1(\bar{\Omega})$ of problem (1)–(3), the energy inequality*

$$\|u\|_{G(t)}^2 + \int_0^t \|u\|_{\Gamma(\tau)}^2 d\tau \leq \text{const} \left\{ \int_0^t [\|f\|_{G(\tau)}^2 + \|g\|_{\Gamma(\tau)}^2] d\tau + \|h\|_G^2 \right\}, \quad 0 \leq t \leq T, \quad (6)$$

holds, where the constant does not depend on t or on $u(t, x)$.

In the proof, approaches proposed for other purposes in (4,8,9) are used. After localization one computes the derivative

$$\frac{d}{dt} \int |R(t)u_\varepsilon(t, x)|^2 dx,$$

where $R(t)$ is a singular integral operator in R_x^n , depending on the parameter t , with symbol, roughly speaking, equal to r , and where $u_\varepsilon(t, x)$ is a smooth

* The derivatives with respect to t, x of order ≤ 2 are continuous in (t, x, ξ) together with their derivatives of arbitrary order with respect to ξ .

continuation of the function $u(t, x)$ to R_x^n , equal to 0 outside the ε -neighborhood of the domain $G(t)$; then one passes to the limit as $\varepsilon \rightarrow 0$.

Denote by $H(\Omega)$, $H^{(q)}(\Omega')$, and $H(G)$ the Hilbert spaces of (vector-) functions in Ω , Ω' , and G , respectively, with norms defined by the equalities

$$\|u\|_{\Omega}^2 = \int_0^T \|u\|_{G(t)}^2 dt, \quad \|g\|_{\Omega'}^2 = \int_0^T \|g\|_{\Gamma(t)}^2 dt,$$

and $\|w\|_G$; in the first and third cases the column vectors have height N , and in the second, height q . Let $f, u \in H(\Omega)$; $g \in H^{(q)}(\Omega')$; $v \in H^{(N)}(\Omega')$; $h, w \in H(G)$. We shall call a triple (u, v, w) a strong solution of problem (1)–(3) if there exists a sequence of functions $u_{\nu}(t, x) \in C^1(\bar{\Omega})$ such that, as $\nu \rightarrow \infty$,

$$\begin{aligned} & \|u_{\nu} - u\|_{\Omega} + \|u_{\nu} - v\|_{\Omega'} + \|u_{\nu}(T, x) - w\|_G + \\ & + \|Lu_{\nu} - f\|_{\Omega} + \|Bu_{\nu} - g\|_{\Omega'} + \|u_{\nu}(0, x) - h\|_G \rightarrow 0. \end{aligned} \quad (7)$$

Thus, for a strong solution, the values on Ω' , $G(0)$, and $G(T)$ are defined.

Theorem 2. If conditions I, II, and IV are satisfied, then for any $f \in H(\Omega)$, $g \in H^{(q)}(\Omega')$, $h \in H(G)$ problem (1)–(3) has a strong solution (u, v, w) . For strong solutions

$$\|u\|_{\Omega} + \|v\|_{\Omega'} + \|w\|_G \leq \text{const}\{\|f\|_{\Omega} + \|g\|_{\Omega'} + \|h\|_G\}, \quad (8)$$

where the constant does not depend on f , g , or h .

The proof of existence is the same as in ⁽¹⁻³⁾: the adjoint problem is used and the theorem on coincidence of strong and weak solutions ⁽²⁾ is applied.

Inequality (8) shows that the strong solution is unique and depends continuously on the right-hand sides.

Remarks. 1°. Theorems 1 and 2 remain valid if, with respect to the boundary condition on Ω' , one assumes, as in ⁽¹⁻³⁾, that it can only be written locally in the form (2). In this case it is convenient to regard $g(t, x)$ as a section of some vector bundle over Ω' with a q -dimensional fiber. The norm $\|g\|_{\Omega'}$ is then defined by means of a partition of unity on Ω' (see, for example, ⁽¹⁰⁾, Sec. 6). The corresponding definition of a strong solution is equivalent to that given in (3).

2°. Suppose that the matrices $A_{\nu}(t, x)$ are Hermitian not only on Ω' , but also near Ω' . Then condition II may be replaced by the condition of constancy of the rank of the matrix $a_{\nu}(t, x)$ on Ω' , and condition III by the condition of nonnegativity of the form $u \cdot a_{\nu} u$ on $\mathfrak{B}(t, x)$. Theorems 1 and 2 remain valid after, for example, the following modifications: $g = 0$; $\|u\|_{\Gamma(\tau)}^2$ on the left in (6) is replaced by $\int_{\Gamma(\tau)} u \cdot a_{\nu} u dx'$; in the definition of a strong solution

$Bu_\nu = 0$. Such theorems are obtained almost directly from the results of $(1-3, 8, 9)$ by localization.

3°. Put, for an integer $l \geq 0$,

$$|||u|||_{G(t),l}^2 = \sum_{\alpha+|\beta|\leq l} \int_{G(t)} \left| \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\beta u}{\partial x^\beta} \right|^2 dx, \quad \|h\|_{G,l}^2 = \sum_{|\beta|\leq l} \int_G \left| \frac{\partial^\beta u}{\partial x^\beta} \right|^2 dx$$

and define similarly $|||u|||_{\Gamma(t),l}$ with the aid of a partition of unity and local coordinates on $\Gamma(t)$. Then, if the smoothness assumptions on the matrices A_ν , B , and C are strengthened by order l , then under conditions I–III one establishes ...

the following inequality, generalizing (6), is established:

$$|||u|||_{G(t),l}^2 + \int_0^t |||u|||_{\Gamma(\tau),l}^2 d\tau \leq \leq \text{const} \left\{ \int_0^t [|||f|||_{G(\tau),l}^2 + |||g|||_{\Gamma(\tau),l}^2] d\tau + \|h\|_{G,l}^2 \right\}, \quad 0 \leq t \leq T. \quad (9)$$

Inequality (9) is proved by means of localization, application of Theorem 1 to the system obtained from (1)–(3) by differentiation, and use of the embedding theorems of N. Aronszajn–L. N. Slobodetskii (see, for example, (11)).

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