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Abstract

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MATHEMATICAL PHYSICS

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A DIFFERENCE METHOD FOR SOLVING CERTAIN COEFFICIENT BOUNDARY-VALUE PROBLEMS

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We shall consider two "coefficient" boundary-value problems (of parabolic type and elliptic type), in which, along with the solution of the differential equation, one of the coefficients of the differential equation is also sought (cf. with ^(1,2)). The existence, uniqueness, and stability of the solution of these problems will be proved, and a difference method for their solution will be justified.

1. Consider the coefficient boundary-value problem for an equation of parabolic type

$$\begin{aligned} a(t)u_{xx}(x, t) - c(t)u(x, t) - u_t(x, t) &= H(x, t), & 0 < x < 1, \\ 0 < t \leq T; \end{aligned} \quad (1,1)$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1; \quad (1,2)$$

$$\begin{aligned} u(0, t) &= f_1(t), & u(1, t) &= f_2(t), & 0 \leq t \leq T, \\ \varphi(0) &= f_1(0), & \varphi(1) &= f_2(0), \end{aligned} \quad (1,3)$$

$$-a(t)u_x(0 + 0, t) = g(t), \quad 0 \leq t \leq T, \quad (1,4)$$

where $c(t) \geq c_{\min} > 0$, $g(t)$, $f_1(t)$, $f_2(t)$, $\varphi(x)$, and $H(x, t)$ are prescribed continuous functions, bounded in their domains of definition, with $\varphi(0) = f_1(0)$, $\varphi(1) = f_2(0)$, while $a(t)$ and $u(x, t)$ are the unknown functions. We note that, for bounded $c(t)$, the fulfillment of the condition $c(t) \geq c_{\min} > 0$ can always be achieved by means of the well-known substitution $u(x, t) = e^{\alpha t}v(x, t)$, taking $\alpha > 0$ sufficiently large.

Definition 1. A pair of functions $a(t)$, $u(x, t)$ will be called a solution of problem (1,1)–(1,4) if 1) $a(t)$ is continuous for $0 \leq t \leq T$, and there exists a constant $a_{\min} > 0$ such that $a(t) \geq a_{\min}$; 2) $u(x, t)$ is defined and continuous in the domain $D = \{0 \leq x \leq 1, 0 \leq t \leq T\}$; 3) $u_x(x, t)$, $u_{xx}(x, t)$, $u_t(x, t)$ are defined and continuous in the domain $D = \{0 < x < 1, 0 < t \leq T\}$.

To find the solution of problem (1,1)–(1,4) we apply a difference-iteration method. Construct a difference mesh by dividing the intervals $0 \leq x \leq 1$ and $0 \leq t \leq T$, respectively, into M and N equal parts by the division points $x_i = ih$, $i = 0, 1, \dots, M$, $x_M = Mh = 1$, $t_j = j\tau$, $j = 0, 1, \dots, N$, $t_N = N\tau = T$. To determine the mesh functions a_j and u_{ij} , which approximate the values of the unknown functions a_j and $u(x_i, t_j)$ at the mesh nodes, we replace problem (1,1)–(1,4) by the difference problem

$$a_j \delta'_{xx} u_{ij} - c_j u_{ij} - \delta_t^- u_{ij} = H_{ij}, \quad 0 < i < M, \quad 0 < j \leq N; \quad (1.5)$$

$$u_{i0} = \varphi_i, \quad 0 \leq i \leq M; \quad (1.6)$$

$$u_{0j} = f_{1j}, \quad u_{Mj} = f_{2j}, \quad 0 \leq j \leq N; \quad (1.7)$$

$$-a_j \delta_x u_{0j} = g_j, \quad 0 \leq j \leq N, \quad (1.8)$$

where $c_j = c(t_j)$, $g_j = g(t_j)$, $f_{1j} = f_1(t_j)$, $f_{2j} = f_2(t_j)$, $H_{ij} = H(x_i, t_j)$, $\varphi_i = \varphi(x_i)$, $f_{10} = \varphi_0$, $f_{20} = \varphi_M$, $\delta_x u_{ij} = h^{-1}(u_{i+1,j} - u_{ij})$, $\delta_t u_{ij} = \tau^{-1}(u_{ij} - u_{i,j-1})$, $\delta_{xx} u_{ij} = h^{-2}(u_{i+1,j} - 2u_{ij} + u_{i-1,j})$.

Assuming that a_m and u_{im} for $0 \leq m \leq j-1$ have already been determined (with $a_0 = -g_0[\delta_x \varphi_0]^{-1}$) and satisfy relations (1.5)–(1.8), to determine a_j and u_{ij} we apply the iteration method (in combination with the sweep method) according to the scheme

$$a_j^{(s)} \delta_{xx} u_{ij}^{(s)} - c_j u_{ij}^{(s)} - \delta_t u_{ij}^{(s)} = H_{ij}, \quad 0 < i < M, \quad 0 < j \leq N; \quad (1.9)$$

$$u_{0j}^{(s)} = f_{1j}, \quad u_{Mj}^{(s)} = f_{2j}, \quad 0 \leq j \leq N; \quad u_{i0} = \varphi_i, \quad 0 < i < M; \quad (1.10)$$

$$-a_j^{(s+1)} \delta_x u_{0j}^{(s)} = g_j, \quad 0 \leq j \leq N, \quad (1.11)$$

where

$$\delta_x u_{0j}^{(s)} = h^{-1}(u_{1j}^{(s)} - u_{0j}^{(s)}), \quad \delta_{xx} u_{ij}^{(s)} = h^{-2}(u_{i+1,j}^{(s)} - 2u_{ij}^{(s)} + u_{i-1,j}^{(s)}), \quad (1.12)$$

$$\delta_t u_{ij}^{(s)} = \tau^{-1}(u_{ij}^{(s)} - u_{i,j-1}^{(s)}).$$

Taking any $a_j^{(0)}$, from (1.9) and (1.10) by the sweep method we find u_{ij}^0 for $0 \leq i \leq M$, and then from (1.11) we find $a_j^{(1)}$, etc.

Theorem 1. Let $H(x, t)$, $\varphi(x)$, $f_1(t)$, $f_2(t)$ be continuous and bounded in their domains of definition; let $H(x, t)$ and $\varphi(x)$ be twice differentiable with respect to x ; let $f_1(t)$, $f_2(t)$ be differentiable with respect to t , and let

$$\delta_{xx} H_{im} \leq 0, \quad \delta_{xx} \varphi_i \geq 0; \quad \delta_x \varphi_i < 0; \quad (1.13)$$

$$c_m f_{1m} + H_{0m} + \delta_t f_{1m} \geq 0, \quad c_m f_{2m} + H_{Mm} + \delta_t f_{2m} \geq 0; \quad (1.14)$$

$$f_{1m} - f_{2m} \geq K_{\min} > 0 \quad (1.15)$$

for $1 \leq m \leq j$, $1 \leq i \leq M-1$. Then the iterative process (1.9)–(1.12) converges for sufficiently large $g_{\min} = \inf g(t) > 0$, and $a_j^{(s)}, u_{ij}^{(s)}$ tend monotonically as $s \rightarrow +\infty$ to their limits a_j, u_{ij} , which form the solution of problem (1.5)–(1.8). For a_j, u_{ij} the estimates

$$1) |u_{ij}| \leq U < +\infty; \quad 2) 0 \leq \delta_{xx} u_{ij} \leq \bar{U} < +\infty; \quad 3) 0 < K_{\min} \leq -\delta_x u_{0j} \leq \bar{\bar{U}} < +\infty;$$

$$4) 0 < a_{\min} \leq a_j \leq a_{\max}, \quad 0 < j \leq N,$$

hold, where the constants $U, \bar{U}, \bar{\bar{U}}, a_{\min}, a_{\max}$ do not depend on h, τ , but depend only on the prescribed functions.

The proof of Theorem 1 is based on the following lemmas.

Lemma 1. If the solution of the difference problem (1.5)–(1.8) exists, then under the conditions of Theorem 1 the estimates

$$1) |u_{ij}| \leq U < +\infty;$$

2*) $0 \leq \delta_{xx} u_{ij} \leq \bar{B}_1/a(h, \tau) + \bar{B}_2$ hold, where the constants $U, \bar{B}_1, \bar{B}_2$ depend only on the prescribed functions, and $a(h, \tau) = \min a_j$ for the values of h, τ under consideration.

Lemma 2. If the solution of the difference problem (1.5)–(1.8) exists and conditions 1) are satisfied, then 3*) $0 < K_{\min} \leq -\delta_x u_{0j} \leq \bar{B}_1/a(h, \tau) + \bar{B}_2^*$, where $\bar{B}_1, a(h, \tau)$ have the same meanings as in Lemma 1, and $\bar{B}_2^*$ is a constant depending only on the prescribed functions.

Lemma 3. If there exists a solution of the difference problem (1.5)–(1.8) and the conditions of Theorem 1 are satisfied, then for sufficiently large $g_{\min} = \inf g(t) > 0$ there exist constants $a_{\min}, a_{\max}$, independent of h, τ , such that the inequalities 4) $0 < a_{\min} \leq a_j \leq a_{\max} < +\infty$ hold for all j , $0 < j \leq N$.

Theorem 2. If the following conditions are satisfied: 1) $H, H_x, H_{xx}, H_{xxx}, H_{xxxx}, H_t, H_{tt}, g, g_t, c, c_t, f_1, f_{1t}, f_{1tt}$ exist and are bounded; 2) $H_{xx} \leq 0$, $f_1(t) - f_2(t) \geq K_{\min} > 0$, $0 \leq c(t)f_1(t) + H(0, t) + f_{1t}(t) \leq \bar{K}$, $0 \leq c(t)f_2(t) + H(1, t) + f_{2t}(t) \leq \bar{K}$, $0 \leq \varphi_{xx}(x) \leq K$, then for sufficiently large $g_{\min} = \inf g(t) > 0$ the solution of the coefficient boundary-value problem (1.1)–(1.4) exists.

The proof of Theorem 2 is based on the estimates of S. N. Bernstein ^(3, 4).

Along with problem (1.1)–(1.4), consider the “perturbed” problem

$$\begin{aligned} \tilde{a}(t)\tilde{u}_{xx}(x, t) - \tilde{c}(t)\tilde{u}(x, t) - \tilde{u}_t(x, t) &= \tilde{H}(x, t), & 0 < x < 1, \\ 0 < t \leq T; \end{aligned} \quad (1.1')$$

$$\tilde{u}(x, 0) = \tilde{\varphi}(x), \quad 0 \leq x \leq 1; \quad (1.2')$$

$$\tilde{u}(0, t) = \tilde{f}_1(t), \quad \tilde{u}(1, t) = \tilde{f}_2(t), \quad 0 \leq t \leq T; \quad (1.3')$$

$$-\tilde{a}(t)\tilde{u}_x(0, t) = \tilde{g}(t), \quad 0 \leq t \leq T, \quad (1.4')$$

where $\tilde{c}(t) \geq c_{\min} > 0$, $\tilde{g}(t) \geq g_{\min} > 0$, $\tilde{f}_1(t)$, $\tilde{f}_2(t)$, $\tilde{\varphi}(x)$, $\tilde{H}(x, t)$ are given functions possessing the same differential properties as the corresponding functions in Theorems 1 and 2 without the tilde. Put

$$\begin{aligned} \delta_1(x) &= \tilde{\varphi}(x) - \varphi(x), & \delta_2(t) &= \tilde{f}_1(t) - f_1(t), & \delta_3(t) &= \tilde{f}_2(t) - f_2(t), \\ \delta_4(t) &= \tilde{g}(t) - g(t), & \delta_5(t) &= \tilde{c}(t) - c(t), & \delta_6(x, t) &= \tilde{H}(x, t) - H(x, t), \\ \alpha(t) &= \tilde{a}(t) - a(t), & \omega(x, t) &= \tilde{u}(x, t) - u(x, t). \end{aligned}$$

Definition 2. Suppose that for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, if

$$|\delta_i| < \delta = \delta(\varepsilon), \quad i = 1, \dots, 6; \quad |\delta_{1x}| < \delta = \delta(\varepsilon), \quad |\delta_{1xx}| < \delta = \delta(\varepsilon)$$

for all x and t in their domains of variation, then for all such x and t the inequalities

$$|\alpha(t)| < \varepsilon, \quad |\omega(x, t)| < \varepsilon$$

hold.

Then we shall say that the solution of problem (1.1)–(1.4) is **stable with respect to small perturbations** of φ, f_1, f_2, g, c , and H .

Theorem 3. *The solution of problem (1.1)–(1.4) under the conditions of Theorem 2 is unique and stable with respect to small perturbations of $\varphi, f_1, f_2, g, c, H$.*

The proof of Theorem 3 is based on estimates of the Green function and its derivative with respect to x for the operator

$$L(u) = a(t)u_{xx} - c(t)u - u_t$$

with boundary conditions of the first kind.

2. Consider the coefficient boundary-value problem for an equation of elliptic type in the simplest (one-dimensional) case:

$$au_{xx}(x) - u(x) = H(x), \quad 0 < x < 1; \quad (2.1)$$

$$u(0) = f_1, \quad u(1) = f_2, \quad f_1 > f_2; \quad (2.2)$$

$$-au_x(0+0) = g, \quad g > 0, \quad (2.3)$$

where $H(x)$ is a given continuous bounded function; f_1, f_2, g are given constants; the constant a and the function $u(x)$ are unknown.

Definition 3. A pair of quantities $\{a, u(x)\}$, where a is a constant and $u(x)$ is a function, will be called a **solution of problem** (2.1)–(2.3) if 1) $a > 0$; 2) $u(x)$ is defined and continuous for $0 \leq x \leq 1$; 3) $u_x(x), u_{xx}(x)$ are defined and continuous for $0 < x < 1$; 4) all relations (2.1)–(2.3) are satisfied.

We shall approximate problem (2.1)–(2.3) by the difference problem

$$a\delta_{xx}u_i - u_i = H_i, \quad 1 \leq i \leq M-1; \quad (2.4)$$

$$u_0 = f_1, \quad u_M = f_2; \quad (2.5)$$

$$-a\delta_x u_0 = g, \quad (2.6)$$

on the difference grid $x_i = ih$, $i = 0, 1, \dots, M$, $Mh = 1$, we shall solve problem (2.4)–(2.6) by means of the iterative scheme

$$a^{(s)} \delta_{\bar{x}x} u_i^{(s)} - u_i^{(s)} = H_i, \quad 0 < i < M; \quad (2.7)$$

$$u_0^{(s)} = f_1, \quad u_M^{(s)} = f_2; \quad (2.8)$$

$$-a^{(s+1)} \delta_x u_0^{(s)} = g. \quad (2.9)$$

Theorem 4. Let $H(x)$ be twice differentiable and let the inequalities

$$-\infty < A \leq \delta_{\bar{x}x} H_i \leq 0, \quad H_0 + f_1 \geq 0, \quad H_M + f_2 \leq 0,$$

$$f_1 - f_2 \geq K > 0. \quad (2.10)$$

be satisfied. Then, for sufficiently large $g > 0$ and for any initial approximation $a^{(0)} > 0$, $a^{(s)}$, $u_i^{(s)}$ tend monotonically as $s \rightarrow +\infty$ to the limits u_i , $a(h)$, which form the solution of the difference problem (2.4)–(2.6), and the estimates

$$0 \leq \delta_{\bar{x}x} u_i \leq \bar{U}, \quad |u_i| \leq U, \quad 0 < K \leq -\delta_x u_i \leq \bar{K},$$

$$0 < a_{\min} < a(h) < a_{\max} < +\infty. \quad (2.11)$$

hold.

Theorem 5. If $H(x)$ is three times differentiable on the interval $0 \leq x \leq 1$, $|H'''(x)| < E < +\infty$, $0 \leq x \leq 1$, then, under the conditions of Theorem 1, the solution of problem (2.1)–(2.3) exists.

Definition 4. We shall say that the solution of problem (2.1)–(2.3) is **stable with respect to small perturbations** $\delta_1 = \tilde{f}_1 - f_1$, $\delta_2 = \tilde{f}_2 - f_2$, $\delta_3(x) = \tilde{H}(x) - H(x)$ and $\delta_4 = \tilde{g} - g$, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, when the conditions

$$|\delta_i| < \delta(\varepsilon), \quad i = 1, 2, 3, 4, \quad 0 \leq x \leq 1 \quad (2.12)$$

are satisfied, the inequalities

$$|\tilde{a} - a| < \varepsilon, \quad |\tilde{u}(x)| < \varepsilon \quad \text{for } 0 \leq x \leq 1 \quad (2.13)$$

will hold.

Theorem 6. Under the conditions of Theorem 5, the solution of problem (2.1)–(2.3) is unique and stable with respect to small perturbations $\delta_1, \delta_2, \delta_3, \delta_4$.

Remark. The proposed difference method can be extended to problems in which, along with the solution of the differential equation, two (or more) coefficients of the differential equation are sought, and also to problems in which the coefficients of the differential equation are unknown functions of x, t, u .

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