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Abstract

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MATHEMATICS

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ON AN INTEGRAL REPRESENTATION OF SMOOTH FUNCTIONS AND SOME FAMILIES OF FUNCTION SPACES

(Presented by Academician S. L. Sobolev on June 11, 1965)

1. We shall assume that e is any subset of the set of natural numbers $e_n = \{1, \dots, n\}$. If $h = (h_1, \dots, h_n)$ is a given vector, then let $h^e = (h_1^e, \dots, h_n^e)$, where $h_j^e = h_j$ for $j \in e$; $h_j^e = 0$ for $j \in e_n \setminus e$. Let $f(x)$ be a function defined in the Euclidean space E^n of points $x = (x_1, \dots, x_n)$; $\Delta_j^{k_j}(t_j)f(x)$ is the finite difference of order k_j with respect to the variable x_j with step t_j . Put

$$\Delta^{k^e}(t)f = \left[\prod_{j \in e} \Delta_j^{k_j}(t_j) \right] f.$$

For each subset $e \subseteq e_n$, $f^{(m^e)}(x)$ is the derivative of the function $f(x)$ of order m^e with respect to the variables x^e . The order of differentiation is arbitrary; for example:

$$f^{(m^e)}(x) = \frac{\partial^{m_1^e}}{\partial x_1^{m_1^e}} \cdots \frac{\partial^{m_n^e}}{\partial x_n^{m_n^e}} f(x).$$

Let

$$\int_0^{h^\sigma} (\cdot) dy^e = \int_0^{h_1^{\sigma_1}} dy_1^e \cdots \int_0^{h_n^{\sigma_n}} (\cdot) dy_n^e,$$

where

$$\int_0^{h_j^{\sigma_j}} dy_j^e = \int_0^{h_j^{\sigma_j}} dy_j \quad \text{for } j \in e,$$

and

$$\int_0^{h_j^{\sigma_j}} dy_j^e$$

is the identity operator for $j \in e_n \setminus e$.

Let Ω and e_n^* be any fixed subsets of the set e_n . Vectors $h = (h_1, \dots, h_n)$ and $\sigma = (\sigma_1, \dots, \sigma_n)$ with positive components are given, and a vector h is such that for $j \in e_n \setminus \Omega$, $h_j = h_0$. For all such vectors an integral representation has been obtained for sufficiently smooth functions f :

$$f^{(\nu)}(x) \equiv \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial^{\nu_n}}{\partial x_n^{\nu_n}} f(x) = \sum_{e \subset \Omega} \sum_{i \in e_n^0 \setminus \Omega} (-1)^{|e|+1} I_e^{(i)}(f). \quad (1)$$

Here $e_n^0 = \{0, 1, \dots, n\}$, $|e|$ is the number of elements of the set e ,

$$\begin{aligned} I_e^{(0)}(f) &\equiv I_e(f, e^*) = \\ &= \int_0^h dv^e \int_0^{v^e} dy^e \int_0^{v^e - y} dt^{e^*} \int_0^{h^e} \Delta^{k^{e^*}}\left(\frac{t}{k}\right) f^{(m^e)}(x + y) R_e dy^{e_n \setminus e}, \end{aligned}$$

where $e^* = e_n^* \cap e$, $m_i + k_i - \nu_i \geq 0$,

$$R_e = R_e(y, t^{e^*}, v^e, h^{e_n \setminus e});$$

$$I_e^{(i)}(f) = \begin{cases} I_{e \cup \{i\}}(f, e^* \cup \{i\}) & \text{for } i \in e_n^*, \\ I_{e \cup \{i\}}(f, e^*) & \text{for } i \in \overline{e_n^*}. \end{cases}$$

The integral kernels R_e are differentiable functions of their arguments and admit an estimate in terms of the components of the vector h .

The integral representation (1) for $e_n^* = e_n$, $\Omega = \emptyset$, i.e., when Ω is the empty set, coincides with the integral representation obtained by V. P. Il' in ⁽¹³⁾.

2. Let $r = (r_1, \dots, r_n)$ be a vector with nonnegative components. By the **supporting vectors** of the vector r we shall mean the smallest subset e of the set e_n such that $r^e \equiv r$. We denote the support of this vector by e_r . Suppose that e_r^* is the set of those indices j from e_r for which, when $j \in e_r^*$, the number r_j is not an integer.

Let e be any subset of the set e_r , and let $e^* = e_r^* \cap e$. For each positive r_j put $r_j = \bar{r}_j + \alpha_j$, where $\bar{r}_j$ is the integer part of r_j , so that $0 \leq \alpha_j < 1$, and if $r_j = 0$, put $\bar{r}_j = 0$. Thus to each vector $r = (r_1, \dots, r_n)$ there corresponds a vector $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$.

Definition. Let e be such a subset of the set e_r that $e^* = \emptyset$, i.e., all r_j ($j \in e$) are integers. We shall say that $f \in L_p^{(r^e)}(E^n)$ if the function $f(x)$ in E^n has a generalized derivative in the sense of S. L. Sobolev ⁽¹⁾, $f^{(r^e)}(x) \in L_p(E^n)$ ($p \geq 1$). The norm in this space is defined as follows:

$$\|f, L_p^{(r^e)}(E^n)\| = \|f^{(r^e)}, L_p(E^n)\| = \left(\int_{E^n} |f^{(r^e)}(x)|^p dE^n \right)^{1/p}.$$

Now let e be such a subset of the set e_r that it intersects the set e_r^* , i.e. the set e^* is nonempty. For such e we shall say that $f \in L_p^{(r^e)}(E^n)$, if $f \in L_p^{(\bar{r}^e)}(E^n)$ and the integral

$$\mathcal{J}_e(f) = \left(\int_0^\infty \dots \int_0^\infty \|\Delta^{\omega e^*}(t)f^{(\bar{r}^e)}, L_p(E^n)\|^p \frac{dt^{e^*}}{\prod_{j \in e^*} t_j^{1+p\alpha_j}} \right)^{1/p}$$

is finite, where $\omega = (1, \dots, 1)$ is the vector all of whose components are equal to one; and we define the norm in this space by

$$\|f, L_p^{(r^e)}(E^n)\| = \|f, L_p^{(\bar{r}^e)}(E^n)\| + \mathcal{J}_e(f).$$

Let Ω be any fixed subset of the set e_r , and let the set e be a subset of the set Ω .

Definition. For $i \in e_r^*$ we shall say that $f \in L_p^{(r^e, r_i)}(E^n)$, if the generalized derivative

$$f^{(r_i)}(x) \equiv \frac{\partial^{r_i}}{\partial x_i^{r_i}} f(x) \in L_p^{(r^e)}(E^n)$$

and we define the norm as follows:

$$\|f, L_p^{(r^e, r_i)}(E^n)\| = \|f^{(r_i)}, L_p^{r^e}(E^n)\|.$$

For $i \in e_r^*$ we shall say that $f \in L_p^{(r^e, r_i)}(E^n)$, if $f \in L_p^{(r^e, \bar{r}_i)}(E^n)$ and the integral

$$\mathcal{J}_e^{(i)}(f) = \left\{ \int_0^\infty [\mathcal{J}_e(\Delta_i(t_i)f^{(\bar{r}_i)})]^{p t_i^{-(1+p\alpha_i)}} dt_i \right\}^{1/p}$$

is finite, and we define the norm by

$$\|f, L_p^{(r^e, r_i)}(E^n)\| = \|f, L_p^{(r^e, \bar{r}_i)}(E^n)\| + \mathcal{J}_e^{(i)}(f).$$

Basic definition. We shall say that $f \in W_p^{(r)}(\Omega, E^n)$, if for every $e \subset \Omega$ and every $i \in e_r^0 \setminus \Omega$, $f \in L_p^{(r^e, r_i)}(E^n)$, where $e_r^0 =$

$= e_r \cup \{0\}$, $r_0 = 0$, and define the norm in this space as follows:

$$\|f, W_p^{(r)}(\Omega, E^n)\| = \sum_{e \subset \Omega} \sum_{i \in e_r^0 \setminus \Omega} \|f, L_p^{(r^e, r_i)}(E^n)\|,$$

where

$$L_p^{(\emptyset,0)}(E^n) \equiv L_p(E^n), \quad L_p^{(r^e,0)}(E^n) \equiv L_p^{(r^e)}(E^n).$$

The space $W_p^{(r)}(\Omega, E^n)$, for $\Omega = \emptyset$ and $e_r = e_n$, coincides with the generalized space of S. L. Sobolev $W_p^{(r_1, \dots, r_n)}(E^n)$ (see ^(1, 5, 7)), and for $\Omega = e_r$ it coincides with the space $S_p^{(r)}W(E^n)$, which for integral r_i ($i \in e_r$) was defined by S. M. Nikol'skii ⁽⁴⁾.

Definition. By the space $W_p^{(r)}(\Omega, E^n)$ we shall mean the closure of the set of smooth finite functions in the norm $\|f, W_p^{(r)}(\Omega, E^n)\|$.

Theorem 1. The spaces $W_p^{(r)}(\Omega, E^n)$ and $W_p^{(r)}(\Omega, E^n)$ coincide for $1 < p < \infty$.

3. Let $r = (r_1, \dots, r_n)$ be a vector with nonnegative components r_i ($i = 1, \dots, n$), whose support is e_r . For each positive r_j put $r_j = \bar{r}_j + \beta_j$, where $\bar{r}_j$ is the greatest integer less than r_j , so that $0 < \beta_j \leq 1$, and when $r_j = 0$ put $\bar{r}_j = 0$; consequently, to each vector $r = (r_1, \dots, r_n)$ there corresponds the vector $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$. Introduce the norms

$$\|f, \mathcal{L}_p^{(r^e)}(E^n)\| = \left(\int_0^\infty \dots \int_0^\infty \|\Delta^{2\omega^e}(t) f^{(\bar{r}^e)}, L_p(E^n)\|^p \frac{dt^e}{\prod_{j \in e} t_j^{1+p\beta_j}} \right)^{1/p},$$

where e is any subset of the set e_r ;

$$\|f, \mathcal{L}_p^{(r^e, r_i)}(E^n)\| = \left(\int_0^\infty \|\Delta_i^2(t_i) f^{(\bar{r}_i)}, \mathcal{L}_p^{(r^e)}(E^n)\|^p \frac{dt_i}{t_i^{1+p\beta_i}} \right)^{1/p},$$

where e is any subset of the set $\Omega \subset e_r$ and $i \in e_r \setminus \Omega$;

$$\|f, B_p^{(r)}(\Omega, E^n)\| = \sum_{e \subset \Omega} \sum_{i \in e_r^0 \setminus \Omega} \|f, \mathcal{L}_p^{(r^e, r_i)}(E^n)\|,$$

where

$$e_r^0 = e_r \cup \{0\}, \quad \mathcal{L}_p^{(\emptyset,0)}(E^n) \equiv L_p(E^n), \quad r_0 = 0.$$

Definition. By the space $B_p^{(r)}(\Omega, E^n)$ we shall mean the closure of the set of sufficiently smooth finite functions in the norm $\|f, B_p^{(r)}(\Omega, E^n)\|$. The space $B_p^{(r)}(\Omega, E^n)$, for $\Omega = \emptyset$ and $e_r = e_n$, coincides with the known space $B_p^{(r_1, \dots, r_n)}(E^n)$, defined and studied by O. V. Besov ⁽⁶⁾ (see also ^(7, 8, 10), etc.),

and for $\Omega = e_r$ it coincides with the known space $S_p^{(r)}B(E^n)$, defined by the author ⁽¹²⁾.

4. Let, for any subset e of the set e_r ,

$$M^{(r^e)}(f) = \sup_{t^e} \left\| \prod_{j \in e} t_j^{-\beta_j} \Delta^{2\omega^e}(t) f^{(\bar{r}^e)}, L_p(E^n) \right\|.$$

For any subset e of the set $\Omega \subset e_r$ and any $i \in e_r \setminus \Omega$ put

$$M^{(r^e, r_i)}(f) = \sup_{t_i} t_i^{-\beta_i} M^{(r^e)}(\Delta_i^2(t_i) f^{(\bar{r}_i)}).$$

Definition. By the space $H_p^{(r)}(\Omega, E^n)$ we shall mean the closure of the set of sufficiently smooth finite functions in the norm

$$\|f, H_p^{(r)}(\Omega, E^n)\| = \sum_{e \subset \Omega} \sum_{i \in e_r^0 \setminus \Omega} M^{(r^e, r_i)}(f),$$

where

$$e_r^0 = e_r \cup \{0\}, \quad M^{(\emptyset, 0)}(f) = \|f, L_p(E^n)\|.$$

An analogous remark is also made for $H_p^{(r)}(\Omega, E^n)$ (see ^(2,3,7,11)).

5. Let the vector $r = (r_1, \dots, r_n)$ be such that $e_r = e_n$, i.e., the components of this vector are positive numbers. Further, let $1 < p \leq q \leq \infty$, let Ω be any fixed subset of the set $e_n = \{1, \dots, n\}$, let m be any natural number $\leq n$ and $e_m = \{1, \dots, m\}$, and let $\nu = (\nu_1, \dots, \nu_n)$ be a vector with nonnegative integer components. Put

$$\varepsilon_j = 1 - \left(\frac{1}{p} - \frac{1}{q} \right) \frac{1}{r_j} - \frac{\nu_j}{r_j} > 0 \quad (j \in \Omega \cap e_m \equiv \Omega^*), \quad \varepsilon_i = 1 - \frac{1}{pr_i} - \frac{\nu_i}{r_i} > 0$$

$$(i \in \Omega \setminus \Omega^*), \quad \varepsilon_\Omega = 1 - \frac{1}{p} \sum_{j \in e_n \setminus \Omega} \frac{1}{r_j} - \sum_{j \in e_n \setminus \Omega} \frac{\nu_j}{r_j} + \frac{1}{q} \sum_{j \in e_m \setminus \Omega^*} \frac{1}{r_j} > 0.$$

Theorem 2. Let $f \in W_p^{(r)}(\Omega, E^n)$, and let the vector $\rho = (\rho_1, \dots, \rho_m)$ be such that its components satisfy the conditions:

- a) for $j \in \Omega^*$, $0 < \rho_j < \varepsilon_j r_j$, if at least one of ρ_j and r_j is an integer;
 $0 < \rho_j \leq \varepsilon_j r_j$, if ρ_j and r_j are simultaneously nonintegers;

- b) for $j \in e_m \setminus \Omega^*$, $0 < \rho_j < \varepsilon_\Omega r_j$, if at least one of ρ_j and r_i ($i \in e_n \setminus \Omega$) is an integer; $0 < \rho_j \leq \varepsilon_\Omega r_j$, if ρ_j and r_i ($i \in e_n \setminus \Omega$) are simultaneously nonintegers.

Then, for any fixed $x_{m+1}, \dots, x_n$, $f^{(\nu)} \in W_q^{(\rho)}(\Omega^*, E^m)$, and the inequality holds

$$\|f^{(\nu)}, W_q^{(\rho)}(\Omega^*, E^m)\| \leq c \|f, W_p^{(r)}(\Omega, E^n)\|,$$

where c is a constant independent of f .

For brevity we shall denote the assertion of this theorem as follows:

$$W_p^{(r)}(\Omega, E^n) \rightarrow W_q^{(\rho)}(\Omega^*, E^m).$$

Theorem 3.

$$B_p^{(r)}(\Omega, E^n) \rightarrow B_q^{(\rho)}(\Omega^*, E^m),$$

where the components of the vector $\rho = (\rho_1, \dots, \rho_m)$ satisfy the conditions: $0 < \rho_j \leq \varepsilon_j r_j$ for $j \in \Omega^*$; $0 < \rho_j \leq \varepsilon_\Omega r_j$ for $j \in e_m \setminus \Omega^*$.

Theorem 4. Under the conditions of Theorem 3,

$$H_p^{(r)}(\Omega, E^n) \rightarrow H_q^{(\rho)}(\Omega^*, E^m).$$

Theorem 5.

$$W_p^{(r)}(\Omega, E^n) \rightarrow B_q^{(\rho)}(\Omega^*, E^m),$$

where the components of the vector $\rho = (\rho_1, \dots, \rho_m)$ satisfy the conditions:

- a) for $j \in \Omega^*$, $0 < \rho_j < \varepsilon_j r_j$, if r_j is an integer; $0 < \rho_j \leq \varepsilon_j r_j$, if r_j is noninteger;
- b) for $j \in e_m \setminus \Omega^*$, $0 < \rho_j < \varepsilon_\Omega r_j$, if at least one of the r_i ($i \in e_n \setminus \Omega$) is an integer; $0 < \rho_j \leq \varepsilon_\Omega r_j$, if all r_i ($i \in e_n \setminus \Omega$) are nonintegers.

Theorem 6. Under the conditions of Theorem 3,

$$B_p^{(r)}(\Omega, E^n) \rightarrow H_q^{(\rho)}(\Omega^*, E^m).$$

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Note: Figure translations are in progress. See original paper for figures.

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