

Maximal and Normal Series of Finite Groups

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Abstract

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MATHEMATICS

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MAXIMAL AND NORMAL SERIES OF FINITE GROUPS

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§ 1. The main results of the present note concern the study of the influence of structural properties of the set of maximal subgroups of a group on its normal structure. Here, by structural properties of one or another set M of subgroups of a given finite group we mean those which are connected with the mutual relations between subgroups from M . These include, for example, the property of permutability of subgroups and the character of their intersections. The normal structure of a group, according to Wielandt, includes properties connected with the factors of normal series of the group. Such are, for example, the properties of solubility and generalized solubility.

In §§ 3 and 4 of the present note we study groups for which certain terms of their maximal series are quasi-subnormal in the group (see the definition below) or permutable. Problems of this kind were considered in papers ^(1,6,10), but there only invariant subgroups were used. The results given below show that subgroups of a more general kind (quasi-subnormal and permutable) in many cases exert the same influence on the normal structure of a group as invariant subgroups. This is confirmed, for example, by assertions 1) and 3) of Theorem 1, and by Theorems 3 and 4. In Theorem 2 one obtains the same class of nonsoluble groups as in ⁽¹⁰⁾, where invariant subgroups are considered.

In § 5 results are given on the influence of maximal subgroups having a given index or a given core on the solubility of a finite group. Theorem 8 generalizes a theorem of Ph. Hall ⁽¹⁶⁾, Theorem 10.5.7, while Theorem 9 generalizes a theorem of B. Huppert ⁽¹⁾, Theorem 9). Particular cases of Theorems 8 and 9 are, respectively, Theorems 10 and 11 of the paper of A. V. Romanovskii ⁽¹⁵⁾.

In § 6 one new property of finite p -soluble groups is given. From Theorem 11, as consequences, one can obtain Theorem 1 of the paper of B. Huppert ⁽¹⁾, as well as a theorem of G. Zappa ⁽¹³⁾ on the indices of maximal subgroups of strongly p -soluble groups.

§ 2. We shall give the notation and definitions used below.

G is a finite group of order (G) ; π is some (empty or not) set of primes; π' is the complement of the set π to the set of all primes; $\pi(G)$ is the set of all primes dividing (G) ; the notation $\pi\sigma\pi'$ means that for any $p \in \pi$ and $q \in \pi'$ one always has $p < q$; $\tau(G)$ is the number of elements of the set $\pi(G)$; $\tau_\pi(G)$ is the number of elements of π belonging to $\pi(G)$; a πd -group is a group whose order is divisible by some prime from π ; E is the identity subgroup of the group G ; $(G)_{\pi'}$ is the greatest π' -divisor ⁽⁵⁾ of the order of G ; S_π^θ is a subgroup of the group G of order $(G)_{\pi'}$ possessing some group-theoretic property θ ; $E_{\pi'}^\theta$ means that G has an S_π^θ -subgroup; group-theoretic properties: d is dispersiveness in the sense of Ore ⁽⁶⁾, z means that all Sylow subgroups of the group are cyclic; the core of a subgroup H is the intersection of all subgroups,

conjugate to H in G ; a group of type S_1 is a nonnilpotent group all of whose proper subgroups are abelian (a Miller–Moreno group). The notions of p -solvability and strong p -solvability are taken by us from (2).

If $G \neq E$, then a series of subgroups

$$G = G_0 \supset G_1 \supset \dots \supset G_n = E, \quad n \geq 1,$$

is called a maximal series of the group G , if each member of this series G_i ($i = 1, 2, \dots, n$) is a maximal subgroup of the preceding member G_{i-1} . If $G = E$, then the unique maximal series of the group G will be taken to be $E = E$.

By the i -th maximal subgroup of the group G ($i \geq 1$) we shall mean the i -th member of some maximal series of it. Γ_i is the set of all i -th maximal subgroups of G ; Γ_1^π is the set of all maximal πd -subgroups of G ; Γ_2^π is the set of all second π -maximal subgroups (14) of the group G .

§ 3. A subgroup H of a group G is called quasireachable if, for any Sylow p -subgroup P of G (p ranges over the entire set $\pi(G)$), the intersection $H \cap P$ is a Sylow subgroup in H (see, for example, (7)). In particular, if (H) is not divisible by p , then the p -Sylow subgroup of H will be taken to be E .

It should be noted that the question remains open whether every quasireachable subgroup of some group G is reachable (subinvariant) in G . The converse assertion, as is not hard to verify, is true (8).

Theorem 1. *If in a group G : 1) all subgroups from the set Γ_1 are quasireachable, or if 2) all subgroups from the set Γ_2 are quasireachable, or if 3) all subgroups from the set Γ_3 are quasireachable, then the group G is, respectively: 1) nilpotent, 2) either nilpotent or a group of type S_1 , 3) solvable, and for $\tau(G) > 3$ even nilpotent.*

Theorem 2. *If in a group G all subgroups from the set Γ_4 are quasireachable, then G will be of one of the following types:*

1. A solvable group.

2. $G \cong SL(2, 5)$.
3. $G \cong LF(2, p)$, where $p = 5$, or p is such a prime number that $p - 1$ and $p + 1$ are products of no more than three (not necessarily distinct) primes, and $p \equiv \pm 3 \pmod{40}$ or $p \equiv \pm 13 \pmod{40}$.

Theorem 2 generalizes Theorems 1 and 2 of (10).

§ 4. Let U and V ($U \neq V$) be certain sets of subgroups of the group G . Then the notation $UV = VU$ means that each subgroup from U permutes with each subgroup from V . We shall further say that the set U is weakly commutative if all subgroups from U that are conjugate to one another in the group G are pairwise permutable.

Theorem 3. *If in a πd -group G the set Γ_1^π is weakly commutative and $\tau_\pi(G) > 1$, then the group G is nilpotent.*

For $\pi(G) \subseteq \pi$, Theorem 3 yields a theorem of O. Ore ((6), Ch. II, § 2, Theorem 1).

Theorem 4. *Let G be a nonnilpotent πd -group and let the set Γ_2^π be weakly commutative. Then G is of one of the following types:*

1. A group of type S_1 .
2. The direct product of a cyclic group of order p by a group of type S_1 , whose order is not divisible by p .

From Theorem 4 one can derive Theorem 23 of (1) and Theorem 23 of (14). At the same time, Theorem 4 sharpens the assertion of Theorem 23 from (14).

Theorem 5. *If in a group G one of the following conditions is satisfied: 1) $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$, 2) $\Gamma_1\Gamma_3 = \Gamma_3\Gamma_1$, 3) $\Gamma_2\Gamma_3 = \Gamma_3\Gamma_2$, then the group G is, respectively: 1) supersolvable, 2 and 3) solvable.*

The following definition was first introduced for consideration by S. A. Chukhikhin in paper (3) (see also (4)).

Definition. A subgroup H of a group G is called π -permutable in G if it is permutable with every p -Sylow subgroup of G for any $p \in \pi$.

Theorem 6. Let H be a proper πd -subgroup of the group G , with $H_G = E$, and let $\tau_\pi(G) > 1$. If H is π -permutable in G , then the group G contains, for each prime divisor $p \in \pi$ of the order of H , a normal divisor of index $p^\alpha n_1$, where $(p, n_1) = 1$ for any $p \in \pi$.

From Theorem 6 there follows one assertion of Theorem 1' of paper (11), establishing an analogous property for a quasinormal (6) subgroup, i.e. for a subgroup that is permutable with all subgroups of the given group. In addition, it follows from Theorem 6 that a simple πd -group for which $\tau_\pi(G) > 1$ cannot contain a proper π -permutable πd -subgroup.

Theorem 7. Let G be a solvable group and

$$G = G_1 G_2 \dots G_k,$$

where $G_1, G_2, \dots, G_k$ are pairwise permutable supersolvable groups whose orders are pairwise relatively prime. If every maximal subgroup of the group G_i is permutable with all G_j , $i, j = 1, 2, \dots, k$, then the group G is supersolvable.

From Theorem 7, using the theorem of H. Wielandt ⁽⁹⁾, we obtain

Corollary. Let

$$G = G_1 G_2 \dots G_k,$$

where $G_1, G_2, \dots, G_k$ are nilpotent groups satisfying all the remaining conditions of Theorem 7. Then the group G is supersolvable.

Theorem 7 and the corollary from it generalize the result of B. Huppert ⁽¹²⁾, p. 164).

§ 5. **Theorem 8.** Let the group G satisfy the following conditions:

1) $\pi \nsubseteq \pi'$, 2) $E_{\pi'}^d$, 3) all maximal subgroups of G containing some of its $S_{\pi'}$ -subgroup H have as their index either a prime number or the square of a prime number. Then the group G is solvable.

Theorem 9. Let the group G satisfy the following conditions:

1) $\pi \nsubseteq \pi'$, 2) $E_{\pi'}^z$. The group G is then and only then supersolvable if all maximal subgroups of G containing some of its $S_{\pi'}^z$ -subgroup H have prime index.

If $\pi(G) \subseteq \pi$, then Theorem 8 becomes Theorem 10.5.7 of P. Hall from ⁽¹⁶⁾, and Theorem 9 becomes Theorem 9 of B. Huppert from ⁽¹⁾. If, however, $\pi(G)$ contains only one prime number from the set π' , then from Theorems 8 and 9 we obtain, respectively, Theorems 10 and 11 of A. V. Romanovskii from ⁽¹⁵⁾.

Theorem 10. Let p be the greatest number in $\pi(G)$. If all maximal subgroups of the group G whose order is divisible by the greatest p -divisor of the order of G have core E , and if every maximal subgroup of G with core E is supersolvable, then the group G is solvable.

§ 6. **Definition 1.** If p^m is the highest power of the number p occurring as an index of all maximal chains of the group G , then $m = m_p(G)$ will be called the maximal p -rank of the group G .

Following B. Huppert ⁽¹⁾, we introduce, for p -solvable groups, the following

Definition 2. Let $p^{\alpha_1}, p^{\alpha_2}, \dots, p^{\alpha_k}$ be all the indices of the chief chains of a p -solvable group G that are powers of the number p . Put

$$\text{Max } \alpha_i = r_p(G),$$

and call $r_p(G)$ the chief p -rank of the group G .

Theorem 11. $r_p(G) = m_p(G)$.

An analogous dependence between chief and maximal chains in finite solvable groups was established earlier by B. Huppert ⁽¹⁾.

§ 7. 1. The example of the tetrahedron group shows that the class of nonnilpotent groups covered by assertion 2 of Theorem 1 is broader than the class of groups all of whose second maximal subgroups are invariant (see ⁽¹⁾, Theorem 23).

2. Using the example of the icosahedron group (with a corresponding choice of the set—

ness of π) one can verify the essentiality of conditions 1 and 3 in Theorem 8, as well as the essentiality of the condition of Theorem 10 requiring that every maximal subgroup of the group with core E be supersolvable.

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