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Abstract

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MATHEMATICS

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AN EXISTENCE THEOREM FOR THE SOLUTION OF THE PROBLEM OF SHORT-RANGE WEATHER FORECASTING

In works ^(1,2) the method of fractional steps (the splitting method, the method of weak approximation) was used to study the correctness of the Cauchy problem for a linear system of differential equations in Banach space. The present note is devoted to proving the existence of a smooth solution of a mixed problem for one quasilinear system of equations. This problem is, in a certain sense (the problem is considered in a domain unbounded in x, y), a model one with respect to the problem used for short-range weather forecasting ⁽³⁾.

Problem A. It is required to determine functions u, v, H of the independent variables x, y, p, t , satisfying the system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - lv + \frac{\partial H}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + lv + \frac{\partial H}{\partial y} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial}{\partial p} \frac{p^2}{\mu} \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial H}{\partial p} &= 0 \end{aligned} \quad (1)$$

for $t > 0$, $0 < p_0 < p < P$, with boundary conditions

$$\begin{aligned} \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial H}{\partial p} &= 0 \quad \text{for } p = p_0, \\ p \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial H}{\partial p} + \alpha \frac{\partial H}{\partial t} &= 0 \quad \text{for } p = P \end{aligned} \quad (2)$$

and initial conditions

$$u(x, y, p, 0) = u_0(x, y, p), \quad v(x, y, p, 0) = v_0(x, y, p),$$

$$H(x, y, p, 0) = H_0(x, y, p). \quad (3)$$

Here l is a smooth function of x, y ; μ is a smooth function of p , with $0 < \mu_0 \leq \mu \leq \mu_1$; $p_0, P, \alpha, \mu_0, \mu_1$ are constants.

Let $\bar{U}(t) = (u, v, H, \partial H / \partial p)$, $D_t = \{S_t \times (p_0, P)\}$; S_t is the section in the (x, y, t) -space by the plane $t = \text{const}$ of the cone with base S_0 and generators inclined in the direction of increasing t at the angle

$$\gamma = \arctg(\varkappa + 1),$$

where

$$\varkappa = C + V_0, \quad V_0 = \max_S \sqrt{u_0^2 + v_0^2},$$

$$C = \max\{\sqrt{4\mu_1}, \sqrt{2\mu(P)/\alpha}\}.$$

Let $W_2^{m, k_0}(D_t)$ be the set of functions having generalized derivatives belonging to $L_2(D_t)$ up to order m , where differentiation with respect to p does not exceed order k_0 .

Definition. We define the norm $\mathbf{U}(t)$ in W_2^{m, k_0} as follows:

$$\begin{aligned} \|\mathbf{U}(t)\|_{W_2^{m, k_0}}^2 = \int_{S_t} \left\{ \sum_{m_1=0}^m \sum_{n+r=m_1} \left(\int_{p_0}^P \left[\left(\frac{\partial^{m_1} u}{\partial x^n \partial y^r} \right)^2 + \left(\frac{\partial^{m_1} v}{\partial x^n \partial y^r} \right)^2 + \frac{p^2}{\mu} \left(\frac{\partial^{m_1+1} H}{\partial x^n \partial y^r \partial p} \right)^2 \right] dp \right. \right. \\ \left. \left. + \frac{\alpha P}{\mu(P)} \left(\frac{\partial^{m_1} H(P)}{\partial x^n \partial y^r} \right)^2 \right) + \sum_{m_1=1}^m \sum_{\substack{1 \leq k \leq k_0 \\ n+r+k=m_1}} \int_{p_0}^P \left[\left(\frac{\partial^{m_1} u}{\partial x^n \partial y^r \partial p^k} \right)^2 + \left(\frac{\partial^{m_1} v}{\partial x^n \partial y^r \partial p^k} \right)^2 \right. \right. \\ \left. \left. + \left(\frac{\partial^{m_1+1} H}{\partial x^n \partial y^r \partial p^{k+1}} \right)^2 \right] dp \right\} dx dy. \end{aligned}$$

Theorem. If $l \in W_2^m(S_0)$, $\mu \in W_2^{k_0}(p_0, P)$, $\mathbf{U}(0) \in W_2^{m, k_0}(D_0)$ for $m \geq 3$, $k_0 \geq 2$, then there exists a $\delta > 0$, depending on $\|\mathbf{U}(0)\|_{W_2^{m, k_0}}$, such that for $0 < t < \delta$ problem A has a solution $\mathbf{U}(t) \in W_2^{m, k_0}(D_t)$.

The uniqueness of the solution of problem A under the conditions of the theorem was proved in [4]. There it was also proved that a solution of problem A exists in the class of functions analytic in x, y, t and admitting majorants uniform in p . By the method of analytic approximation, in [5] the existence of a smooth solution of problem A was proved. In the present paper we prove the theorem by another method. This method was proposed in [3, 6, 8] for the numerical solution of the short-range weather-forecast problem.

The solution $\mathbf{U}(t)$ of problem A is constructed as the limit of solutions $\mathbf{U}_\tau(t)$ of problems A_τ as τ tends to zero. Problem A_τ “weakly approximates” problem A and consists in the successive solution of the mixed problem A' for system (1') and the mixed problem A'' for system (1'').

Problem A_τ . The vector-function $\mathbf{U}_\tau(t)$ is continuous in t ; $\mathbf{U}_\tau(0) = \mathbf{U}(0)$; for $2(n-1)\tau < t \leq (2n-1)\tau$ (n natural) it satisfies the system of equations

$$\begin{aligned} \frac{\partial u_\tau}{\partial t} + 2 \frac{\partial H_\tau}{\partial x} &= 0, \\ \frac{\partial v_\tau}{\partial t} + 2 \frac{\partial H_\tau}{\partial y} &= 0, \\ 2 \left(\frac{\partial u_\tau}{\partial x} + \frac{\partial v_\tau}{\partial y} \right) - \frac{\partial}{\partial p} \frac{p^2}{\mu} \frac{\partial^2 H_\tau}{\partial p \partial t} &= 0 \end{aligned} \quad (1')$$

and the boundary conditions

$$\begin{aligned} \frac{\partial^2 H_\tau}{\partial p \partial t} &= 0 \quad \text{for } p = p_0, \\ p \frac{\partial^2 H_\tau}{\partial p \partial t} + \alpha \frac{\partial H_\tau}{\partial t} &= 0 \quad \text{for } p = P; \end{aligned} \quad (2')$$

for $(2n-1)\tau < t \leq 2n\tau$ it satisfies the system of equations

$$\begin{aligned} \frac{\partial u_\tau}{\partial t} + a_\tau \frac{\partial u_\tau}{\partial x} + b_\tau \frac{\partial u_\tau}{\partial y} - 2lv_\tau &= 0, \\ \frac{\partial v_\tau}{\partial t} + a_\tau \frac{\partial v_\tau}{\partial x} + b_\tau \frac{\partial v_\tau}{\partial y} + 2lu_\tau &= 0, \\ \frac{\partial^2 H_\tau}{\partial t \partial p} + a_\tau \frac{\partial^2 H_\tau}{\partial x \partial p} + b_\tau \frac{\partial^2 H_\tau}{\partial y \partial p} &= 0 \end{aligned} \quad (1'')$$

and the boundary condition

$$\frac{\partial H}{\partial t} = 0 \quad \text{for } p = P, \quad (2'')$$

where

$$a_\tau \equiv 2u_\tau((2n-1)\tau), \quad b_\tau \equiv 2v_\tau((2n-1)\tau).$$

The solution of problem A_τ is reduced to the successive solution of two very simple problems A' and A'' (problem A' , for example, can be solved by the Fourier method). Relying on the embedding theorem of S. L. Sobolev ⁽⁷⁾, one

can show that both of these problems are uniformly well posed, in the sense of definition ⁽¹⁾, in $W_2^{m,k_0}(D_t)$.

The reserve of smoothness ($m \geq 3$, $k_0 \geq 2$) ensures compactness of the set $\{U_\tau(t)\}$ for t less than some δ , determined by the initial data, and the existence of a limiting vector-function $U(t) \in W_2^{m,k_0}(D_t)$, which is the limit of $U_{\tau_k}(t)$ as $\tau_k \rightarrow 0$. If $U_\tau(t)$ is substituted into the equations of system (1) and the boundary conditions (2), then they will be satisfied with an error equal to right-hand sides that weakly approximate zero. Hence it follows that $U(t)$ is a solution of problem A, and, by virtue of the uniqueness of the solution of problem A, it follows that $U^\tau(t)$ converges to $U(t)$ as τ tends to zero.

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