

ON THREE LINEARIZED INVERSE PROBLEMS FOR HYPERBOLIC EQUATIONS

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Abstract

Full Text

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MATHEMATICS

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ON THREE LINEARIZED INVERSE PROBLEMS FOR HYPERBOLIC EQUATIONS

(Presented by Academician S. L. Sobolev on 19 III 1966)

1. Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = n^{-2} \Delta u, \quad (1)$$

where n is a function of the variables x, y ; Δ is the Laplace operator with respect to these same variables. Let $u(x, x_0, y, t)$ be a generalized solution of equation (1) in the half-plane $y > 0$, satisfying the following boundary and initial conditions:

$$\begin{aligned} u(x, x_0, y, 0) &= \frac{\partial}{\partial t} u(x, x_0, y, 0) = 0, \\ \frac{\partial}{\partial y} u(x, x_0, 0, t) &= \delta(x - x_0) \delta(t), \end{aligned} \quad (2)$$

where $\delta(t)$ is the Dirac delta function. Further, let $\tau(x_1, x_2)$ be the largest of the numbers τ such that the support of the function $u(x_1, x_0, 0, t)$, for given x_1, x_0 , is contained in the half-line*

$$\tau \leq t < \infty.$$

For equation (1) we pose the following inverse problem:

Problem 1. It is required, from the function $\tau(x_1, x_0)$, to determine the function $n(x, y)$.

In the case where the function n does not depend on the variable x , the formulated inverse problem has been studied in the works ^(1,2).

The function $\tau(x_1, x_0)$ is the minimum of the functional

$$I(\Gamma) = \int_{\Gamma} n(x, y) ds,$$

where Γ is a curve joining the points $(x_1, 0)$, $(x_0, 0)$ and lying in the upper half-plane.

Let now

$$n(x, y) = n_0(x, y) + n_1(x, y),$$

where the function $n_0(x, y)$ is given, and the function $n_1(x, y)$ is small. Correspondingly, the function $\tau(x_1, x_0)$ is represented in the form of the sum

$$\tau(x_1, x_0) = \tau_0(x_1, x_0) + \tau_1(x_1, x_0).$$

Here the function $\tau_0(x_1, x_0)$ corresponds to the function $n_0(x, y)$.

Neglecting quantities of order $n_1^2(x, y)$, we obtain that the function $\tau_1(x_1, x_0)$ is represented in the form

$$\tau_1(x_1, x_0) = \int_{\Gamma(x_1, x_0)} n_1(x, y) ds, \quad (3)$$

* In the process described by equation (1), $\tau(x_1, x_0)$ is the time in which a disturbance produced at the point

where $\Gamma(x_1, x_0)$ is the curve joining the points x_1, x_0 , for which the minimum of the functional is attained

$$I(\Gamma) = \int_{\Gamma} n_0(x, y) ds.$$

We investigate the posed linearized problem for one special case of the function $n_0(x, y)$. Let

$$n_0(x, y) = (ay + b)^{-1},$$

where $a > 0$, $b \geq 0$.

In this case (see (3)) the curve $\Gamma(x_1, x_0)$ is an arc of a circle with center at the point $P((x_1 + x_0)/2, -b/a)$. Let us extend the functions $n_0(x, y)$ and $n_1(x, y)$ into the lower half-plane in an even manner. In this case the posed problem reduces to the problem of determining the function $n_1(x, y)$ from the equation

$$\tau_1(x - r, x + r) = \frac{r}{2} \int_0^{2\pi} n_1(x + r \cos \varphi, r \sin \varphi) d\varphi. \quad (4)$$

2. Denote by $u(M, M_0, t)$ the generalized solution of the equation

$$\partial^2 u / \partial t^2 = \Delta u + a(M)u + \delta(M - M_0)\delta(t) \quad (5)$$

in the domain $z > 0$ under the following initial and boundary conditions:

$$u(M, M_0, 0) = \frac{\partial}{\partial t} u(M, M_0, 0) = 0, \quad (6)$$

$$\frac{\partial}{\partial z} u(M_1, M_0, t) = 0. \quad (7)$$

Here $M(x, y, z)$, $M_0(x_0, y_0, 0)$, $M_1(x, y, 0)$ are points of three-dimensional space; Δu is the Laplace operator with respect to the variables x, y, z .

Assuming that the function $a(M)$ is small, consider the problem of reconstructing it from the solution of the problem (5), (6), (7) known in the plane $z = 0$.

It is convenient to reduce the posed problem to an equivalent problem for the whole space, extending the function $a(M)$ into the half-space $z < 0$ in an even manner. Represent the solution $u(M, M_0, t)$ in the form

$$u(M, M_0, t) = u_0(M, M_0, t) + u_1(M, M_0, t),$$

where the function $u_0(M, M_0, t)$ satisfies the wave equation

$$\partial^2 u / \partial t^2 = \Delta u + \delta(M - M_0)\delta(t)$$

and the conditions (6), (7). As is known, the function $u_0(M, M_0, t)$ has the form

$$u_0(M, M_0, t) = \delta(t - r(M, M_0)) / 2\pi r(M, M_0). \quad (8)$$

In this formula $r(M, M_0)$ is the distance between the points M and M_0 .

For the function $u_1(M, M_0, t)$ we obtain, neglecting the term of order $a(M)u_1(M, M_0, t)$, the linearized equation

$$\partial^2 u_1 / \partial t^2 = \Delta u_1 + a(M)u_0(M, M_0, t) \quad (9)$$

and zero initial data. Taking formula (8) into account, we find that the function $u_1(M, M_0, t)$ has the form

$$u_1(M, M_0, t) = -\frac{1}{8\pi^2 \sqrt{t^2 - r^2(M, M_0)}} \iint_S \frac{a(P) ds}{\sqrt{r(P, M_0) r(P, M)}}, \quad (10)$$

where $P(\xi, \eta, \zeta)$ is the variable point of integration, S is the surface of the ellipsoid

$$r(P, M_0) + r(P, M) = t.$$

From formula (10) follow two possible formulations of the problem of reconstructing the function $a(M)$.

Problem II. The point M_0 is fixed, and the function $u_1(M_1, M_0, t) = \varphi(M_1, M_0, t)$ is known. From formula (10) it follows in this case that the integral equation of the first kind is

$$\frac{1}{8\pi^2} \iint_S \frac{a(P) ds}{\sqrt{r(P, M_1) r(P, M_0)}} = -\sqrt{t^2 - r^2(M_1, M_0)} \varphi(M_1, M_0, t). \quad (11)$$

Problem III. The point M_0 is a variable point, and the function $u_1(M_0, M_0, t) = \psi(M_0, t)$ is known. In this case, letting the point M tend to the point M_0 in formula (10), we obtain the equation

$$\frac{1}{4\pi^2 t^2} \iint_S a(P) ds = -\psi(M_0, t), \quad (12)$$

in which the surface of integration is a sphere of radius $t/2$ with center at the point M_0 . Consequently, we arrive at the problem of reconstructing a function even in z from the known values of its integrals over arbitrary spheres whose centers lie in the plane $z = 0$.

3. The uniqueness of the solution of Problems I and III follows directly from the results contained in the book of R. Courant ⁽⁴⁾.

The uniqueness of the solution of Problem II can be proved by a method analogous to that contained in this book.

Problems I, II, and III are ill-posed in the sense of Hadamard, but well-posed in the sense of A. N. Tikhonov.

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CITED LITERATURE

- ¹ G. Herglotz, *Zs. Math. u. Phys.*, **52**, 275 (1905).
- ² E. Wiechert, *Nachr. Gesellschaft Wiss. Göttingen*, 415 (1907).
- ³ V. I. Smirnov, *A Course of Higher Mathematics*, 4, Moscow, 1957.
- ⁴ R. Courant, *Partial Differential Equations*, "Mir," 1964.

Note: Figure translations are in progress. See original paper for figures.

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