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Abstract

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MATHEMATICS

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ON SOME PROPERTIES OF ALGEBRAS OF CONTINUOUS FUNCTIONS

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§ 1. Basic definitions. In this note the following notation and terms will be used: X is a compact Hausdorff space; $C(X)$ is the algebra of all complex continuous functions on X with the uniform norm; $M(X)$ is the space of all finite complex Borel measures on X ; a measure is an element of the space $M(X)$; a partition is a covering of the space X by pairwise disjoint closed sets; $A \subset C(X)$ — A is a closed subalgebra of the algebra $C(X)$ containing the constants; $f|F$ is the restriction of the function $f \in C(X)$ to the set F ; $A|F$ is the algebra of all restrictions to F of functions from $A \subset C(X)$; $(A|F)^-$ is the uniform closure of the algebra $A|F$; $\operatorname{Re} f$ is the real part of the function f ; $\operatorname{Re} A = \{\operatorname{Re} f : f \in A\}$; χ_F is the characteristic function of the set F .

If $A \subset C(X)$ and $F \subset X$, then F is called a set of antisymmetry (for A) if every real-valued function in $A|F$ is constant; F is a peak set (for A) if there exists a function $f \in A$ such that $f|F = 1$ and $|f(x)| < 1$ for $x \in X \setminus F$. A closed set $F \subset X$ will be called a set of weak analyticity (for A) if every peak set for the algebra $(A|F)^- \subset C(F)$ either coincides with F or is nowhere dense in F .

If $A \subset C(X)$ and $\mu \in M(X)$, then $\mu \perp A$ means that $\int f d\mu = 0$ for all $f \in A$.

§ 2. The spaces H^∞ . Let $A \subset C(X)$. For every positive measure $\mu \in M(X)$, denote by $H^1(\mu)$ the closure of the algebra A in the space $L^1(\mu)$, and by $H^\infty(\mu)$ the intersection $H^1(\mu) \cap L^\infty(\mu)$. Put

$$\operatorname{Re} H^\infty(\mu) = \{\operatorname{Re} f : f \in H^\infty(\mu)\}.$$

Lemma 1. (1) $H^\infty(\mu)$ is a closed subalgebra of the algebra $L^\infty(\mu)$; (2) from every bounded sequence $f_n \in H^\infty(\mu)$ in $L^\infty(\mu)$ one can extract a subsequence converging weakly in $L^1(\mu)$ to some function $f \in H^\infty(\mu)$; (3) if $F \subset X$ and $\chi_F \in \operatorname{Re} H^\infty(\mu)$, then $\chi_F \in H^\infty(\mu)$.

We shall call a positive measure μ antisymmetric if every real-valued function in $H^\infty(\mu)$ is constant almost everywhere with respect to μ .

Lemma 2. (1) If μ is an antisymmetric measure, then the support $\text{Sp } \mu$ of the measure μ is a set of weak analyticity; (2) every set of weak analyticity is a set of antisymmetry.

Lemma 3. If $f \in C(X)$ and $f|_{\text{Sp } \mu} \in A|_{\text{Sp } \mu}$ for every antisymmetric measure μ , then $f \in A$.

§ 3. Restoring partitions. Let $A \subset C(X)$. A partition $\mathcal{K}$ of the space X will be called restoring if: (1) from the conditions $f \in C(X)$ and $f|_K \in A|_K$ for each $K \in \mathcal{K}$ it follows that $f \in A$; (2) for each maximal ideal I of the algebra A there exists a unique $K \in \mathcal{K}$ for which $(I|_K)^-$ is a maximal ideal of the algebra $(A|_K)^-$.

Example 1. The Shilov partition $\mathcal{K}_0$. Points $x_1, x_2 \in X$ belong to one and the same $K \in \mathcal{K}_0$ if and only if $f(x_1) = f(x_2)$ for every real-valued function $f \in A$ (see (1), § 44).

Example 2. The Bishop partition $\mathcal{K}_1$. Its elements are the maximal, with respect to inclusion, antisymmetry sets (see $(^2, ^3)$).

The partitions $\mathcal{K}_0$ and $\mathcal{K}_1$ are recovering and, moreover, satisfy the condition: for every $K \in \mathcal{K}_i$ ($i = 0, 1$) the algebra $A|_K$ is closed in $C(K)$. Here we shall show that there exist recovering partitions finer than the Bishop partition.

Lemma 4. Let $\mathcal{R}$ be some collection of subsets of the space X . Among all partitions $\mathcal{K}$ satisfying the condition: every $R \in \mathcal{R}$ is contained in some $K \in \mathcal{K}$, there exists a finest one. We shall denote this partition by $\mathcal{K}(\mathcal{R})$.

Theorem 1. A. Let $A \subset C(X)$. Then: 1) every set of weak analyticity is contained in a maximal one, and the collection $\mathcal{R}$ of all maximal sets of weak analyticity covers X ; (2) if $f \in C(X)$ and $f|_R \in A|_R$ for all $R \in \mathcal{R}$, then $f \in A$; (3) the partition $\mathcal{K}_2 = \mathcal{K}(\mathcal{R})$ is recovering; 4) every element of the partition $\mathcal{K}_2$ is contained in some element of the Bishop partition $\mathcal{K}_1$.

B. If X is a metrizable space and $A \subset C(X)$, then for every $K \in \mathcal{K}_2$ the algebra $A|_K$ is closed in $C(K)$.

C. If $A_1 \subset C(X)$, $A_2 \subset C(X)$ and the partitions $\mathcal{K}_2$ coincide for the algebras A_1 and A_2 , then the partitions $\mathcal{K}_1$ also coincide for them.

Proof. Assertion A(1) is verified with the aid of Zorn's lemma; A(2) follows from Lemmas 2(1) and 3. Assertions A(3) and B, as well as Lemma 3, are proved by the methods developed in $(^3)$ (see pp. 416-417, 419, 421-422). Assertion A(4) follows from Lemma 2(2). To prove assertion C, note that if f is a real-valued function from A_1 , then it is constant on every $K \in \mathcal{K}_2$ and, consequently, belongs to A_2 . Therefore the partition $\mathcal{K}_0$ is uniquely determined by the partition $\mathcal{K}_2$. Applying the same reasoning to every element of the partition $\mathcal{K}_0$ and continuing thus by transfinite induction, we construct the partition $\mathcal{K}_1$.

We give an example showing that the partition $\mathcal{K}_2$ may be substantially finer than the partition $\mathcal{K}_1$.

Example. Let X_1 be the prism $0 \leq x \leq 1, 0 \leq t \leq 1, |y| \leq t$ in three-dimensional arithmetic space with coordinates x, y, t ; let A_1 be the algebra of all functions defined and continuous on X_1 and analytic in the interior of each section $t = \text{const}$. Denote by X the compactum obtained by identifying the points $(s, 0, 0) \in X_1$ with the points $(0, 0, s) \in X_1$ for all $s \in (0, 1]$. Let φ be the natural projection of X_1 onto X , and let

$$A = \{f : f \in C(X); f \circ \varphi \in A_1\}$$

($f \circ \varphi$ denotes the superposition of the mapping φ and the function f). The algebra A separates the points of X , is closed in $C(X)$, and is antisymmetric. At the same time the partition $\mathcal{K}_2$ for the algebra A is nontrivial and consists of the layers $t = \text{const}$ ($0 < t \leq 1$) and the point $(0, 0, 0)$. Moreover, X is the space of maximal ideals of the algebra A .

§ 4. The space $\text{Re } A$. In this paragraph we establish the connection between the partitions $\mathcal{K}_0$ and $\mathcal{K}_1$ and the space $\text{Re } A$ of real parts of functions from the algebra A . First note (see ⁽⁴⁾) that $\text{Re } A$ may be considered as a real Banach space with norm N :

$$N(u) = \inf\{\|u + iv\| : u + iv \in A\} \quad (u \in \text{Re } A).$$

Denote by C_R the set of all continuous real-valued functions defined on the real line.

Theorem 2. *Let $u \in \text{Re } A$. In order that $u \in A$, it is necessary and sufficient that $f \circ u \in \text{Re } A$ for all $f \in C_R$, where $f \circ u$ denotes the function defined on X :*

$$(f \circ u)(x) = f(u(x)).$$

Proof. Necessity follows directly from the Stone-Weierstrass theorem.

Let $f \circ u \in \text{Re } A$ for all $f \in C_R$, and let μ be an antisymmetric measure (see § 2) and F the support of the measure μ . We shall show that $u|_F = \text{const}$. If this is not so, then there is a bounded sequence of functions $p_n = f_n \circ u$ ($f_n \in C_R$) in $C(X)$, converging pointwise to the characteristic function χ_G of some closed set $G \subset X$, whose intersection with F is different from F and has interior points in F . Clearly,

$$0 < \mu(G) < \mu(F).$$

But on the subspace $S = \{f \circ u : f \in C_R\}$ of the space $\text{Re } A$, the norm N , as is easily checked, is equivalent to the uniform norm on X . Therefore the sequence p_n is bounded in the norm N , and there exists a sequence $g_n \in A$, bounded in $C(X)$, such that $\text{Re } g_n = p_n$. Applying Lemma 1 (2) to the sequence g_n and then Lemma 1 (3), we obtain that

$$\chi_G \in H^\infty(\mu).$$

But this is impossible, since μ is an antisymmetric measure.

Thus the function u is constant on the supports of all antisymmetric measures. Since the algebra A contains the constants, by Lemma 3 we conclude that $u \in A$.

Corollary. If $A_1 \subset C(X)$, $A_2 \subset C(X)$, and $\text{Re } A_1 = \text{Re } A_2$, then the decompositions $\mathfrak{K}_0$ and $\mathfrak{K}_1$ for the algebras A_1 and A_2 coincide.

Glicksberg ⁽³⁾ proved that if a closed G_δ -set F is such that the condition $\mu \perp A$ ($\mu \in M(X)$) implies $\chi_F \mu \perp A$, then F is a peak set for A . Using this result and arguing in essentially the same way as in the proof of Theorem 2, one can prove that the peak sets are determined by the space $\text{Re } A$.

Theorem 3. Let $A_1 \subset C(X)$ and $A_2 \subset C(X)$. If $\text{Re } A_1 = \text{Re } A_2$, then every peak set for A_1 is a peak set for A_2 .

We shall state the last theorem without proof. It relates the space $\text{Re } A$ to the Gleason parts ⁽⁵⁾ and to the decomposition $\mathfrak{K}_0$. Let $A \subset C(X)$. For arbitrary $x, y \in X$, put

$$\rho_A(x, y) = \sup\{|f(x) - f(y)| : f \in A, \|f\| = 1\}.$$

As Gleason showed, any two sets of the form

$$V_x = \{y : y \in X, \rho_A(x, y) < 2\}$$

(such sets are called parts) either coincide or are disjoint. For each set $F \subset X$, the number

$$D_A(F) = \sup\{\rho_A(x, y) : x, y \in F\}$$

will be called the diameter of the set F .

Now let $A_1 \subset C(X)$ and $A_2 \subset C(X)$. Denote by $(\text{Re } A_1)^-$ the uniform closure of the space $\text{Re } A_1$, and by $\mathfrak{K}_0(A_2)$ the Shilov decomposition for the algebra A_2 .

Theorem 4. Let $A_1 \subset A_2$. In order that $(\text{Re } A_1)^- \subset \text{Re } A_2$, it is necessary and sufficient that there exist a number c , $0 \leq c < 2$, such that

$$D_{A_1}(K) \leq c$$

for every $K \in \mathfrak{K}_0(A_2)$.

Corollary 1. If A_2 is an antisymmetric algebra and $(\text{Re } A_1)^- \subset \text{Re } A_2$, then A_1 consists only of constants (this follows from the fact that $D_{A_1}(X)$ can be equal only to either 0 or 2).

Corollary 2 (Hoffman-Wermer theorem ⁽⁶⁾). If an algebra $A \subset C(X)$ separates points and $\text{Re } A$ is closed in the topology of uniform convergence, then $A = C(X)$.

Indeed, put $A_1 = A_2 = A$ in Theorem 4. Then every $K \in \mathfrak{K}_0(A_2)$ consists of a single point (otherwise $D_{A_1}(K) = 2$). Therefore $A = C(X)$.

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