

# A SUBGROUP CHARACTERIZATION OF SOME FINITE GROUPS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## A SUBGROUP CHARACTERIZATION OF SOME FINITE GROUPS

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§ 1. What can be said about a group if information is known about some of its subgroups? This may be information about the structure of these subgroups or about the way in which they are embedded in the containing group. Questions of this kind are considered in many recent works on the theory of finite groups. The principal results formulated in the present note belong to this direction of the theory of finite groups.

In Theorems 1 and 2, for the first time, as far as we know, an exact conclusion is drawn about the structure of a group from information about the structure of its  $n$ -th maximal subgroups (here  $n$  may be any natural number). Theorems 3 and 4 generalize a number of the author's previous results <sup>(1-3)</sup>. Theorem 5 may be regarded as a generalization of a known result, proved independently by S. Janko and by the author, on groups with nilpotent second maximal subgroups. Theorems 6 and 7 generalize a number of the main results of the papers <sup>(2-4)</sup>. In Theorem 8 a result of R. Armstrong <sup>(5)</sup> is extended to nonsolvable groups. In Theorem 9 the nonabelian simple groups are described whose order contains no more than 7 prime factors (the analogous result was known for simple groups whose order contains no more than 5 prime factors). A special case of Theorem 10 is the complete description of nonsolvable groups with abelian third maximal subgroups. Theorem 11 generalizes the main result of the paper <sup>(6)</sup>.

For convenience we give the basic definitions and notation used in this work (we do not indicate where and by whom the corresponding concept or notation was first introduced, referring for information to the papers <sup>(1-3)</sup>).

$|G|$  is the order of the group  $G$  (only finite groups are considered);

$$\lambda(n) = \sum_1^k \alpha_i, \quad \tau(n) = k, \quad \text{if } n = \prod_1^k p_i^{\alpha_i}$$

is the factorization of the natural number  $n$  into prime factors;  $p, q, r$  are prime numbers only;  $G_p$  is a Sylow  $p$ -subgroup of the group  $G$ ;  $H_G$  is the intersection of all subgroups conjugate to  $H$  in  $G$ ;  $H/H_G$  is the cofactor of the subgroup  $H$  in

the group  $G$ ; a  $p$ -decomposable group is a group of the form  $G = G_p \times H$ ; an  $A_p$ -group is a  $p$ -decomposable group with abelian  $p$ -Sylow subgroup; a  $Z_p$ -group is a group with cyclic  $p$ -Sylow subgroup; a group of type  $S$  is a minimal nonnilpotent group; two subgroups are nonincident if neither of them is contained in the other; an  $IC_2^s$ -group is a group in which the intersection of any two nonincident solvable subgroups is a  $Z_2$ -group;  $L'_2(p^n)$  is the factor group of the general linear group  $GL(2, p^n)$  by its center;  $M_9$  is the stabilizer of a point in the Mathieu group  $M_{11}$ ;  $\lambda(G) = \lambda(|G|)$ ;  $\tau(G) = \tau(|G|)$ .

A subgroup  $H$  of the group  $G$  will be called a maximal solvable subgroup if  $H \neq G$  and  $H$  is not contained in any other proper solvable

subgroup of the group  $G$ ; the set of all such  $H$  will be denoted by the symbol  $\Delta_1$ . A  $k$ -th maximal solvable subgroup of the group  $G$ ,  $k > 1$ , is any  $(k-1)$ -st maximal subgroup of a subgroup from  $\Delta_1$ ; the set of all  $k$ -th maximal solvable subgroups of the group  $G$  will be denoted by the symbol  $\Delta_k$ .

Let

$$C: \quad G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_n$$

be some maximal chain of subgroups of the group  $G$ , and let  $s$  be the number of members of the chain  $C$  attainable in  $G$ . We shall consider only such chains  $C$  beginning with  $G$  for which  $s > 0$ . Put  $v(C) = n/s$ . The maximum  $\max[v(C)]$ , where  $C$  runs through all maximal chains of subgroups of the group  $G$  under consideration, will be denoted by the symbol  $v(G)$  and called, following Deskins, the variance of the group  $G$ .

A nonnilpotent group  $G$  will be called a group of type  $D_0$  if it is solvable and all its proper normal divisors are nilpotent. A group of type  $D_0$  will be called a group of type  $D_r$  if all its Sylow subgroups of odd order are regular in the sense of P. Hall.

§ 2. In this section the results obtained are presented.

**Theorem 1.** *If all  $n$ -th maximal subgroups of a group  $G$  are of type  $S$ , then  $n = 1$  and  $G$  is isomorphic to  $LF(2, 5)$  or  $SL(2, 5)$ .*

**Theorem 1a.** *If all subgroups from  $\Delta_n$  are of type  $S$ , then the same conclusion as in Theorem 1 holds.*

**Theorem 2.** *Suppose all  $n$ -th maximal subgroups of the group  $G$  are Frobenius groups; then  $G$  either is itself a Frobenius group, or is isomorphic to one of the groups  $LF(2, 2^p)$ ,  $Sz(2^p)$  (the last two groups occur only when  $n = 1$ ).*

**Theorem 3.** *Suppose every nonnilpotent maximal subgroup of a nonsolvable group  $G$  is a dihedral group, an octahedral group, a Frobenius group, or of type  $D_0$ . Then  $G$  is isomorphic to one of the following groups:  $SL(2, 5)$ ;  $LF(2, 2^p)$ ;  $LF(2, 3^p)$ ,  $p > 2$ ;  $Sz(2^p)$ ;  $LF(2, p)$ ,  $p^2 \equiv -1 \pmod{5}$ ; any nonsolvable Frobenius group whose complement is nonmaximal in it.*

**Theorem 4.** *If in a nonsolvable group  $G$  every nonnilpotent solvable subgroup is a dihedral group, an octahedral group, a Frobenius group, or of type  $D_0$ , then  $G$  is isomorphic to one of the following groups:  $LF(2, p^n)$ ;  $Sz(2^{2n+1})$ ;  $SL(2, 5)$ ; a Frobenius group with complement  $SL(2, 5)$ ; an extension of  $LF(2, p^n)$ ,  $p > 2$ , by means of a group of order 2 (however, the last group does not always satisfy the condition).*

**Lemma.** *If a nonsolvable group  $G$  contains a maximal subgroup with core  $L$  and all maximal subgroups with core  $L$  are nilpotent or of type  $D_r$ , then  $G$  is isomorphic to  $SL(2, 5)$  or  $LF(2, 2^p)$ , where  $2^p - 1$  is a Mersenne prime.*

**Theorem 5.** *Let the cofactors of all maximal subgroups of the group  $G$  be nilpotent or of type  $D_r$ , and let  $R$  be the solvable radical of the group  $G$ . Then either  $G = R$ , or*

$$G/R = L_1 \times L_2 \times \cdots \times L_n,$$

where  $L_i \cong LF(2, p_i + 1)$ , and the  $p_i$  are pairwise distinct Mersenne primes,  $i = 1, 2, \dots, n$ .

In Theorem 6 the number  $m$  denotes that one of the numbers  $p^n - 1$ ,  $p^n + 1$  which is divisible by 4.

**Theorem 6.** *Suppose that in a nonsolvable group  $G$  all subgroups from  $\Delta_4$  are  $Z\alpha$ -groups. Then it is isomorphic to one of the groups:*

- a)  $LF(2, p^n)$  and  $SL(2, p^n)$ ,  $p > 2$  and  $\lambda(m) < 5$ ;
- b) an extension of  $LF(2, p^n)$ ,  $p > 2$ ,  $p^n \neq 7$  and  $\lambda(m) < 4$ , by means of a group of prime order; an extension of a group of prime order by means of  $LF(2, p^n)$ ,  $p^n$  the same as above; an extension of the identity subgroup of order 2 by means of  $L'_2(7)$ ;
- c)  $L'_2(7)$ ,  $LF(2, 8)$ ,  $M_9$ ;
- d) a Frobenius group with complement  $SL(2, 5)$  and an invariant multiplier of order  $q^2$ ;
- e) an extension of  $SL(2, p^n)$ ,  $p^n$  as in b), by means of a group of order  $q$ .

*If  $q = 2$ , then  $G_2$  is the generalized quaternion group.*

From Theorem 6 one easily obtains a description of the nonsolvable groups in which every solvable subgroup  $H$  has the property  $\lambda(H) < 5$ .

**Theorem 7.** *Let the variation  $v(G)$  of a nonsolvable group  $G$  be less than 6. Then  $G$  is isomorphic to one of the following groups:  $LF(2, p^n)$ ,  $p^n = 8, 9, 27$ ;  $LF(2, p)$ ,  $\lambda(p \pm 1) < 5$ ;  $SL(2, p)$ ,  $\lambda(p \pm 1) < 4$ ;  $P \times L$ ,  $|P| = p$ ,  $v(L) = 4$  ( $^2$ ); an extension of  $SL(2, p)$ ,  $p = 5, 7$ , by a group of order 2, where  $G_2$  is the generalized quaternion group;  $L'_2(p)$ ,  $\lambda(p \pm 1) < 4$ .*

We do not formulate the description, readily following from Theorem 7, of nonsolvable groups with invariant fifth maximal subgroups (we note that the variation of such groups does not exceed five).

We shall say that  $G$  is a  $(C)$ -group if any two of its subgroups of the same order are isomorphic.

**Theorem 8.** *Let in a nonsolvable group  $G$  every nonnilpotent biprimary subgroup of even order be a  $(C)$ -group. Then  $G$  contains such a Hall normal divisor  $L$  that either  $G/L \simeq SL(2, 5)$ , or  $L \simeq LF(2, 2^n)$ .*

**Theorem 9.** *Let  $G$  be a nonabelian simple group with  $\lambda(G) < 8$ . Then  $G$  is isomorphic to  $A_7$  or  $LF(2, p^n)$  for suitable  $p$  and  $n$ .*

**Theorem 10.** *If in a nonsolvable group  $G$  all subgroups from  $\Delta_3$  are  $A_2$ -groups, then it is isomorphic to one of the following groups:*

- a)  $LF(2, 2^p)$ ,  $\lambda(2^p \pm 1) < 3$ ; if  $\lambda(2^p - 1) = 2$ , then  $2^p - 1 = qr$ ,  $q \neq r$ , and  $p$  is an exponent of 2 modulo  $q$  and modulo  $r$ ;
- b)  $LF(2, 3^p)$  and  $SL(2, 3^p)$ ,  $p > 2$  and  $\lambda(3^p \pm 1) < 4$ ;
- c)  $LF(2, p)$  and  $SL(2, p)$ ,  $\lambda(p \pm 1) < 4$ ; for  $SL(2, p)$ ,  $p \neq 7$ ;
- d)  $S_5$ ,  $L'_2(7)$ ,  $D \times LF(2, 5)$ ,  $|D| = p$ .

We do not give the description, following from Theorem 10, of nonsolvable groups with abelian third maximal subgroups.

**Theorem 11.** *Let in a nonsolvable group  $G$  all solvable subgroups have the property  $IC_2^s$ . Then  $G$  is isomorphic to one of the groups listed below:*

- a)  $LF(2, 5)$ ,  $LF(2, 8)$ ,  $SL(2, 5)$ ;
- b)  $LF(2, 3^p)$ ,  $SL(2, 3^p)$ ,  $LF(2, 5^p)$ ,  $SL(2, 5^p)$ , where each of the numbers  $3^p + 1$ ,  $5^p - 1$  is the fourth power of an odd prime number;
- c)  $LF(2, p)$  and  $SL(2, p)$ , where that one of the numbers  $p - 1$ ,  $p + 1$  which is divisible by 4 is the fourth power of an odd prime number.

**Theorem 12.** *If a group is not generated by its third maximal subgroups, then it is solvable.*

**Theorem 13.** *Let  $M$  be a normal divisor of a  $p$ -group  $G$ . If  $M$  contains no subgroups of type  $(p, p)$  invariant in  $G$ , then  $M$  is one of the following groups:*

- a) a cyclic group;
- b) a generalized quaternion group;
- c) a dihedral group;
- d)  $G = \{a, b \mid a^2 = b^{2^{n-1}} = 1, aba = b^{-1+2^{n-2}}, n > 3\}$ .

For  $M = G$  one obtains Roquette's theorem <sup>(7)</sup>. For  $M \subseteq \Phi(G)$  one obtains Ch. Hobby's theorem on the Frattini subgroup of a  $p$ -group <sup>(8)</sup>.

Let  $Z_i(G)$  denote the  $i$ -th term of the upper central series of the group  $G$ .

**Corollary 1.** *If  $G$  is a nonabelian  $p$ -group and the subgroup  $Z_2(G)$  is cyclic, then  $G$  is one of the groups b)–d) of Theorem 13, and  $|G| > 8$ .*

In Lemma 1.4 of the work <sup>(9)</sup> it is shown that under the hypotheses of Corollary 1 the number  $p = 2$ .

**Corollary 2.** *Let  $H$  be a cyclic subgroup of a noncyclic  $p$ -group  $G$  such that every cyclic subgroup nonincident with it intersects  $H$  in 1. Then either  $|H| = p$ , or  $G$  is a dihedral group.*

**Corollary 3.** *Let all characteristic abelian subgroups*

*of an abelian  $p$ -group  $G$  are cyclic. Then the subgroup  $\Phi(G)$  is cyclic; moreover, if  $p > 2$ , then  $|(G, G)| = p$ .*

**Theorem 14.** Suppose that any two  $n$ -maximal subgroups of a  $p$ -group  $G$  have a cyclic intersection. Then one of the following assertions is true:

- a)  $\lambda(G) < n + 3$ ;
- b)  $G$  contains a cyclic subgroup of index  $p$ ;
- c)  $G$  is an extension of the quaternion group  $R$  by a generalized quaternion group of order  $2^{n+1}$  (this is possible only for  $n > 1$ );  $R \subset \Phi(G)$ , the center of  $G$  is cyclic and has order 4,  $\Phi(G)$  coincides with the commutator subgroup, and  $G$  contains only one noncyclic  $n$ -maximal subgroup (this subgroup has two independent generators of orders 2 and 4 and is abelian).

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