

EXTREMAL PROBLEMS IN THE THEORY OF POLYNOMIAL OPERATIONS

D. L. Berman

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.88672>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.88:513.83+517.948.35+517.948.5:518

MATHEMATICS

D. L. BERMAN

EXTREMAL PROBLEMS IN THE THEORY OF POLYNOMIAL OPERATIONS

(Presented by Academician S. N. Bernstein on December 9, 1965)

1°. Let us introduce notation. Π_n is the set of all trigonometric polynomials of order $\leq n$. $\tilde{L}$ is the set of all summable 2π -periodic functions. E is a linear normed functional space having the following properties: 1) the elements of E are functions from $\tilde{L}$; 2) if $f \in E$, then the shifted function $f_t(x) = f(x+t)$, for any $-\infty < t < \infty$, also belongs to E , and moreover $\|f_t\| \leq \|f\|$; 3) E contains the set of all trigonometric polynomials. The most important special cases of the space E are: the space $\tilde{C}$ of all continuous 2π -periodic functions, and the space $\tilde{L}_r$ of all 2π -periodic functions summable to the r -th power. To the polynomial

$$\Phi(t) = \sum_{k=0}^n r_k \sin(kt + \alpha_k)$$

we assign the polynomial

$$\tilde{\Phi}(t) = r_n + 2 \sum_{k=0}^{n-1} r_k \cos[(n-k)t + \alpha_n - \alpha_k],$$

which we shall call associated (after F. Riesz (1)) with the polynomial Φ . Put

$$\sigma(f, x) = \int_0^{2\pi} f(x+t) \Phi(t) dt.$$

Denote by $\Omega_n^\Phi(E)$ the set of all linear operations U from E into E having the property that $U(t_n) = \sigma(t_n)$, if $t_n \in \Pi_n$. We also introduce the set $\Omega_{n,n+m}^\Phi(E)$, $m \geq 0$, consisting of all linear operations U from E into E for which the following conditions are satisfied: 1) for every $f \in E$, $U(f) \in \Pi_{n+m}$; 2) if $t_n \in \Pi_n$, then $U(t_n) = \sigma(t_n)$. Introduce the quantities

$$\rho_n = \rho_n^\Phi(E) = \inf_{U \in \Omega_n^\Phi(E)} \|U\|; \quad \rho_{n,n+m} = \rho_{n,n+m}^\Phi(E) = \inf_{U \in \Omega_{n,n+m}^\Phi(E)} \|U\|. \quad (1)$$

Since $\Omega_{n,n+m}^\Phi \subset \Omega_n^\Phi$, it follows that $\rho_{n,n+m} \geq \rho_n$, $m \geq 0$. The present note is devoted to the study of the quantities (1).

2°. Theorem 1. If

$$\tilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right) \geq 0, \quad \varphi_r = \frac{2r-1}{2n}\pi, \quad r = 1, 2, \dots, 2n, \quad (2)$$

then $\rho_n = \pi r_n$.

In this case the extremal operation is given by the equality

$$\bar{U}(f, x) = \sum_{r=1}^{2n} \lambda_r f\left(x + \varphi_r - \frac{\alpha_n}{n}\right); \quad \lambda_r = (-1)^{r-1} \frac{\pi}{2n} \tilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right), \quad (3)$$

i.e. $\|\bar{U}\| = \rho_n$.

We outline the proof. It is known from (2) that for any $t \in \Pi_n$, $\bar{U} = \sigma(t)$.

Consequently, $\bar{U} \in \Omega_n^\Phi$. Since the inequalities (2) hold, by virtue of (3) we have

$$\|\bar{U}(f)\| \leq \|f\| \frac{\pi}{2n} \sum_{r=1}^{2n} \tilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right). \quad (4)$$

But it is easy to verify that

$$\frac{1}{2n} \sum_{r=1}^{2n} \tilde{\Phi}\left(\varphi_r - \frac{\alpha_n}{n}\right) = r_n.$$

Therefore (4) takes the form

$$\|\bar{U}(f)\| \leq \pi r_n \|f\| \quad \text{or} \quad \|\bar{U}\| \leq \pi r_n. \quad (5)$$

Hence

$$\rho_n \leq \pi r_n. \quad (6)$$

On the other hand, in [3] it was proved that

$$\rho_n \geq \pi r_n. \quad (7)$$

From (6) and (7) it follows that

$$\rho_n = \pi r_n. \quad (8)$$

Since $\|\overline{U}\| \geq \rho_n$, by virtue of (8) we have

$$\|\overline{U}\| \geq \pi r_n. \quad (9)$$

The inequalities (5) and (9) lead to the conclusion that

$$\|\overline{U}\| = \pi r_n. \quad (10)$$

Finally, from (8) and (10) it follows that $\|\overline{U}\| = \rho_n = \pi r_n$. Theorem 1 strengthens the lemma from [3].

3°. We now consider the quantity $\rho_{n,2n}^\Phi$ for the case when $E = \widetilde{C}$ or $E = L$. In [4, 5] the following theorem was proved.

Theorem. In the space $\widetilde{C}$ or L , $\rho_{n,n+m}^\Phi$ satisfies the equality

$$\rho_{n,n+m}^\Phi = \inf_{\alpha_k, \beta_k} = I(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_m),$$

where

$$\begin{aligned} I(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_m) = \\ = \int_0^{2\pi} \left| \Phi(t) + \sum_{j=n+1}^{m+n} (\alpha_{j-n} \cos jt + \beta_{j-n} \sin jt) \right| dt. \end{aligned} \quad (11)$$

If the integral (11) attains its minimum for $\alpha_j = \alpha_j^{(0)}$, $\beta_j = \beta_j^{(0)}$, $j = 1, 2, \dots, m$, then the extremal operation is computed by the formula

$$\overline{U}(f, x) = \int_0^{2\pi} f(x+t) \left[\Phi(t) + \sum_{j=n+1}^{m+n} (\alpha_{j-n}^{(0)} \cos jt + \beta_{j-n}^{(0)} \sin jt) \right] dt, \quad (12)$$

i.e. $\|\overline{U}\| = \rho_{n,n+m}^\Phi$.

This theorem may be formulated in the following equivalent form:

Theorem 2. In order that the operator (12) have the smallest norm in the class of operators $\Omega_{n,n+m}^\Phi$, it is necessary and sufficient that the equalities

$$\int_0^{2\pi} \text{sign} \left[\Phi(t) + \sum_{j=n+1}^{n+m} (\alpha_{j-n}^{(0)} \cos jt + \beta_{j-n}^{(0)} \sin jt) \right] e^{i(n+k)t} dt = 0, \quad 1 \leq k \leq m. \quad (13)$$

hold. In this case

$$\|\Omega_{n,1n+m}^\Phi\| = \int_0^{2\pi} \left| \Phi(t) + \sum_{j=n+1}^{n+m} (\alpha_{j-n}^{(0)} \cos jt + \beta_{j-n}^{(0)} \sin jt) \right| dt.$$

Theorem 2 makes it possible, for one important case, to compute the quantity $\rho_{n,2n}$.

Theorem 3. If the kernel Φ is such that

$$\Phi(t) \geq 0, \quad -\infty < t < \infty, \quad (14)$$

then

$$\rho_{n,2n} = 4r_n. \quad (15)$$

Proof. Put

$$\psi(t) = \sum_{k=0}^{n-1} r_k \sin[(2n-k)t + 2\alpha_n - \alpha_k].$$

It is not hard to see that

$$\Phi(t) + \psi(t) = \sin(nt + \alpha_n) \tilde{\Phi}(t).$$

Since (14) holds, we have

$$\int_0^{2\pi} \text{sign}(\Phi(t) + \psi(t)) e^{i(n+k)t} dt = \int_0^{2\pi} \text{sign} \sin(nt + \alpha_n) e^{i(n+k)t} dt. \quad (16)$$

It is known that

$$\text{sign} \sin x = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}.$$

Therefore from (16) we conclude that

$$\int_0^{2\pi} \operatorname{sign}(\Phi(t) + \psi(t)) e^{i(n+k)t} dt = 0, \quad k = 1, 2, \dots, n.$$

Thus, the equalities (13) are satisfied. Consequently, by Theorem 2, the operator

$$\overline{U}(f, x) = \int_0^{2\pi} f(x+t) [\Phi(t) + \psi(t)] dt$$

has the smallest norm in the class $\Omega_{n,2n}^\Phi$, and

$$\rho_{n,2n}^\Phi = \int_0^{2\pi} |\sin(nt + \alpha_n)| \widetilde{\Phi}(t) dt.$$

From this, by a simple calculation, we obtain (15).

Corollary. If the associated kernel is nonnegative, then in the spaces $\widetilde{C}$ and $\widetilde{L}$ the equality

$$\rho_{n,2n}^\Phi : \rho_n^\Phi = 4 : \pi$$

holds.

This assertion follows directly from Theorems 1 and 3.

Theorem 4. Let

$$\Phi(t) = \sum_{k=1}^n b_{n-k} \cos kt,$$

for

$$\begin{aligned} b_\nu - 2b_{\nu+1} + b_{\nu+2} &\geq 0, \quad \nu = 0, 1, 2, \dots, (n-3), \\ b_{n-2} - 2b_{n-1} &\geq 0, \quad b_{n-1} \geq 0. \end{aligned} \quad (17)$$

Then the extremal operator $\overline{U}$ from the class $\Omega_{n,2n-1}^\Phi$ is determined by the equality

$$\overline{U}(f, x) = \int_0^{2\pi} f(x+t) \cos nt \widetilde{\Phi}(t) dt, \quad \widetilde{\Phi}(t) = b_0 + \sum_{j=1}^{n-1} 2b_j \cos jt, \quad (18)$$

where $\rho_{n,2n-1}^\Phi = 4b_0$.

This theorem follows from Theorem 3 and L. Fejér's theorem⁶, according to which the polynomial (18) is nonnegative if its coefficients satisfy inequalities (17).

Remark. Theorem 4 remains valid also in the case when

$$\Phi(t) = \sum_{k=1}^n b_{n-k} \sin kt.$$

Let us apply Theorem 4 in the case when

$$\Phi(t) = \frac{1}{\pi} \left(\frac{\sin(n + \frac{1}{2})t}{2 \sin t/2} \right)'. \quad (19)$$

Then

$$\tilde{\Phi}(t) = \frac{\sin^2 nt/2}{\pi \sin^2 t/2}.$$

Consequently, $\rho_{n,2n-1}^\Phi = 4n : \pi$. It is known¹ that the operator (12), when $\Phi(t)$ is defined according to (19), has the property that for any $t \in \Pi_n$, $\sigma(t) = t'$. Therefore we have

Theorem 5. *Among all linear operations from $\tilde{C}$ to $\tilde{C}$ (from $\tilde{L}$ to $\tilde{L}$) that take all functions into trigonometric polynomials of order $\leq (2n-1)$ and have the property that each polynomial $t \in \Pi_n$ is taken into its derivative, the operator*

$$\overline{U}(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) \sin nt \tilde{\Phi}(t) dt$$

has the smallest norm.

Here $\|U\| = \rho_{n,2n-1}^\Phi = 4n : \pi$.

Hence, in particular, it is clear that with the aid of operators from $\Phi_{n,2n-1}^\Phi$ it is impossible to prove S. N. Bernstein's classical theorem on the modulus of the derivative of a trigonometric polynomial with the sharp constant equal to 1. The best constant that can be obtained by operators from $\Omega_{n,2n-1}^\Phi$, as Theorem 5 shows, is $4 : \pi$.

In conclusion we formulate one more problem, which is solved with the aid of Theorem 3. Let $\mathfrak{M}_n$ be the set of all polynomials of the form $\Phi(t) = \cos t + a_2 \cos 2t + \dots + a_n \cos nt$, for which the associated polynomials are nonnegative on the number axis. It is required to compute the quantity

$$\tau_n = \inf_{\Phi \in \mathfrak{M}_n} \rho_{n,2n-1}^\Phi(E),$$

where $E = \tilde{C}$ or $E = \tilde{L}$, and to find the extremal polynomial. The solution of this problem is given by Theorem 6.

Theorem 6. *For every $n \geq 2$, $\tau_n = 8$. The extremal polynomial has the form $\Phi(t) = \cos t + 2 \cos nt$.*

Leningrad Institute of Soviet Trade
named after Fr. Engels

Received
29 X 1965

References

- ¹ F. Riesz, C. R., 158 (1914).
- ² D. L. Berman, DAN, 163, No. 3 (1965).
- ³ D. L. Berman, DAN, 161, No. 5 (1965).
- ⁴ D. L. Berman, DAN, 138, No. 4 (1961).
- ⁵ D. L. Berman, Matem. sborn., 60 (102), No. 3 (1963).
- ⁶ L. Fejér, Acta Lit. Sci. Univ. Hung., 2 (1925).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.