

HYPERSCALES OF BANACH STRUCTURES

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Abstract

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MATHEMATICS

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HYPERSCALES OF BANACH STRUCTURES

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In this paper we shall use the terminology and notation of article (1).

1. Let Ω be some set with a finite normalized measure m . Denote by $\mathcal{A}$ the topological vector space consisting of all functions $x(t)$ measurable with respect to the measure m ($t \in \Omega$), with the topology of convergence in measure. In the space $\mathcal{A}$ the notion of semi-order is introduced naturally: $x \leq y$ when $x(t) \leq y(t)$ for almost all $t \in \Omega$.

A Banach space $E \subset \mathcal{A}$, which together with every function $x(t)$ contains the function $|x(t)|$, and for which from the inequality $|x(t)| \leq |y(t)|$ it follows that $\|x\|_E \leq \|y\|_E$, is called a **functional Banach structure** (see (2)). By an **ideal structure** we shall mean a functional Banach structure for which the following condition is satisfied: if $y(t) \in E$, $x(t) \in \mathcal{A}$, and $|x(t)| \leq |y(t)|$, then $x(t) \in E$.

Theorem 1. Let $x(t), z(t)$ be elements of an ideal structure E . Then the function $|x(t)|^\theta |z(t)|^{1-\theta}$ ($0 \leq \theta \leq 1$) belongs to E , and

$$\| |x(t)|^\theta |z(t)|^{1-\theta} \|_E \leq \|x\|_E^\theta \|z\|_E^{1-\theta}.$$

The operator $A(x, z) = |x|^\theta |z|^{1-\theta}$ is continuous from $E \times E$ into E .

In what follows we shall everywhere use the following notation:

$$\mu = (\gamma - \beta)/(\gamma - \alpha), \quad \nu = (\beta - \alpha)/(\gamma - \alpha).$$

Definition 1. A continuous normal scale (see (1)) of ideal structures H_α ($0 \leq \alpha \leq 1$), possessing the property

$$\| |x(t)|^\mu |z(t)|^\nu \|_{H_\beta} \leq \|x\|_{H_\alpha}^\mu \|z\|_{H_\gamma}^\nu, \quad (0 \leq \alpha \leq \beta \leq \gamma \leq 1) \quad (1)$$

for all $x, z \in H_1$, is called a **hyperscale**.

We note that it is sufficient to verify inequality (1) only on a dense set of the space H_1 .

If the ideal structures H_0 and H_1 are related Banach spaces, then the minimal scale constructed from the spaces H_0 and H_1 (see (1)) is a hyperscale.

It is not difficult to prove that among all hyperscales joining the ideal structures H_0 and H_1 there exists a hyperscale $H_\alpha^{\max}$ ($0 \leq \alpha \leq 1$) possessing the following property: if H_α ($0 \leq \alpha \leq 1$) is an arbitrary hyperscale joining the spaces H_0 and H_1 , then

$$\|x\|_{H_\alpha} \leq \|x\|_{H_\alpha^{\max}} \quad (x \in H_1).$$

The hyperscale $H_\alpha^{\max}$ ($0 \leq \alpha \leq 1$) will be called **maximal**. It turns out that the maximal hyperscale joining the spaces $\mathcal{L}_{p_0}(\Omega)$ and $\mathcal{L}_{p_1}(\Omega)$ ($1 \leq p_0 \leq p_1 \leq \infty$) is the scale of spaces $\mathcal{L}_p(\Omega)$, where

$$p = p_0 p_1 / [p_1 - \alpha(p_1 - p_0)], \quad 0 \leq \alpha \leq 1.$$

2. An operator T acting from a Banach structure E into the topological space $\mathcal{A}$ is called **convex on the cone of nonnegative functions** if for all $x_1, x_2 \in K$ and $\lambda \in [0, 1]$

$$T(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda T(x_1) + (1 - \lambda)T(x_2).$$

Theorem 2. Let $H_\alpha^{\max}$ ($0 \leq \alpha \leq 1$) be a maximal hyperscale, F_α ($0 \leq \alpha \leq 1$) an arbitrary hyperscale, and let T be a homogeneous, monotone convex operator on the cone K , acting from the space $H_i^{\max}$ into F_i ($i = 0, 1$).

If for the operator T

$$\|T\|_{H_i^{\max} \rightarrow F_i} = M_i < \infty \quad (i = 0, 1),$$

then it maps the space $H_\alpha^{\max}$ into F_α and

$$\|Tx\|_{F_\alpha} \leq M_0^{1-\alpha} M_1^\alpha \|x\|_{H_\alpha^{\max}} \quad (x \in H_\alpha^{\max}).$$

An analogous theorem can be proved for a family of linear integral operators

$$T_\alpha(x) = \int_\Omega K_\alpha(t, s)x(s) dm,$$

where $K_\alpha(t, s) \geq 0$ (almost everywhere) and, for almost all $t, s \in \Omega$, $K_\alpha(t, s)$ is logarithmically convex in α .

3. Denote by $E(\alpha)$ ($0 < \alpha \leq 1$) the set of all x from the ideal structure E with unit $e(t) \equiv 1$, for which the functional

$$\Phi_\alpha(x) = \| |x|^{1/\alpha} \|_E^\alpha < \infty.$$

The functional $\Phi_\alpha(x)$ is a norm defined on $E(\alpha)$, and $E(\alpha)$ is complete with respect to the norm $\Phi_\alpha(x)$.

Using the results of A. P. Calderón⁽³⁾, one can show that the space $E(\alpha)$ is reflexive if the space E is reflexive.

We note that in the case of KB -spaces the norm $\Phi_\alpha(x)$ was considered earlier in⁽⁴⁾.

It is not difficult to see that for any bounded function

$$\lim_{\alpha \rightarrow 0^+} \| |x|^{1/\alpha} \|_E^\alpha = \|x\|_{\mathcal{L}_\infty(\Omega)},$$

therefore it is natural to take $E(0) = \mathcal{L}_\infty(\Omega)$.

The generalized Hölder inequality holds:

$$\|xy\|_E \leq \|x\|_{E(\alpha)} \|y\|_{E(1-\alpha)} \quad (x \in E(\alpha), y \in E(1-\alpha)).$$

If the space $\mathcal{L}_\infty(\Omega)$ is normally embedded (see (1)) in E , then the spaces $E(1-\alpha)$ form a maximal hyperscale joining E and $\mathcal{L}_\infty(\Omega)$.

4. A. P. Calderón (see⁽³⁾) proposed the following construction of intermediate spaces K_α for ideal structures H_0 and H_1 :

$$\|x\|_{K_\alpha} = \inf_{\|x_0\|_{H_0} \leq 1, \|x_1\|_{H_1} \leq 1} \text{vrai max}_{t \in \Omega} \frac{|x(t)|}{|x_0(t)|^{1-\alpha} |x_1(t)|^\alpha}.$$

If $H_1 = \mathcal{L}_\infty(\Omega)$ and $H_0 = E$, then $K_\alpha = E(1-\alpha)$.

Theorem 3. Let K_θ ($0 \leq \theta \leq 1$) be Calderón's family of intermediate spaces joining the ideal structures H_0 and H_1 . For every interval $[\alpha_0, \beta_0] \subset [0, 1]$ the space $\bar{K}_\alpha$ of Calderón's family $\bar{K}_\alpha$ ($0 \leq \alpha \leq 1$), joining the spaces K_{α_0} and K_{β_0} , coincides with the space K_θ , $\theta = \alpha_0(1-\alpha) + \beta_0\alpha$, and has the same norm.

It can be shown that for any two related ideal structures H_0 and H_1 , Calderón's spaces K_α form a hyperscale on the interval $[0, 1]$. This hyperscale is proper (see⁽¹⁾).

Theorem 4. Let H_0 and H_1 be related ideal structures, and suppose that the set M of all bounded measurable functions is dense in the space H_1 . Then Calderón's scale K_α ($0 \leq \alpha \leq 1$) coincides with the maximal hyperscale constructed from the spaces H_0 and H_1 .

Corollary. Let H_α ($0 \leq \alpha \leq 1$) be an arbitrary hyperscale connecting two ideal structures H_0 and H_1 , for which the conditions of Theorem 4 are satisfied. If $x_0 \in H_0$, $x_1 \in H_1$, and $0 \leq \alpha \leq 1$, then the function

$$x(t) = |x_0(t)|^{1-\alpha} |x_1(t)|^\alpha$$

belongs to the space H_α .

5. Let E_α ($0 \leq \alpha \leq 1$) be a continuous normal scale of ideal structures connecting the ideal structures H_0 and H_1 . Suppose that the set of all bounded measurable functions is dense in H_1 .

The family of Banach spaces $E_\alpha(1/p)$ ($p \geq 1$) will be called the **power transform** of the scale E_α ($0 \leq \alpha \leq 1$). It is easy to see that $E_\alpha(1/p)$ is a continuous normal scale.

Theorem 5. Let E_0 and E_1 be related ideal structures, and let the set M be dense in E_1 . Then the minimal scale constructed from the spaces $E_0(1/p)$ and $E_1(1/p)$ majorizes the scale $E_\alpha^{\min}(1/p)$ (see (1)).

We give an example of an interpolation theorem that is a consequence of Theorem 5.

Consider the minimal scale $E_\alpha^{\min}$, connecting the spaces $\mathcal{L}_1$ and $\mathcal{L}_\infty$, which is the Marcinkiewicz scale M_α^0 , consisting of all functions $x(t) \in \mathcal{A}$ for which

$$\lim_{\text{mes } e \rightarrow 0} \left[\int_e |x(t)| dt / (\text{mes } e)^\alpha \right] = 0$$

with norm

$$\|x\|_{M_\alpha^0} = \sup_{\text{mes } e \neq 0} \left[\int_e |x(t)| dt / (\text{mes } e)^\alpha \right] < \infty$$

(see (1)). Every continuous normal scale F_α ($0 \leq \alpha \leq 1$), majorizing the minimal scale $F_\alpha^{\min}$ ($0 \leq \alpha \leq 1$), possesses the normal interpolation property with respect to the power transform of the Marcinkiewicz scale $M_\alpha^0(1/p)$.

Let us note that the maximal scale connecting two related ideal structures consists of ideal structures.

Theorem 6. The power transform of the maximal scale connecting two related ideal structures H_0 and H_1 such that the set M is dense in H_1 is a regular scale and, consequently, possesses the strict interpolation property with respect to every minimal scale.

As an example one may give the power transform of the Lorentz scale Λ_α (see (5)), which is the maximal scale connecting the spaces $\mathcal{L}_1$ and $\mathcal{L}_\infty$ (see (1)).

The space conjugate to $\Lambda_\alpha(1/p)$ consists of measurable functions $f(t)$ for which

$$\|f\|_{\Lambda_\alpha(1/p)'} = \inf \left\{ (1-\alpha) \int_0^1 t^{-\alpha} D(t)^q dt \right\}^{1/q} < \infty,$$

where the infimum is taken over all functions $D(t)$ such that

$$(1 - \alpha) \int_0^s t^{-\alpha} D(t) dt \geq \int_0^s f^*(t) dt$$

for all $s \in [0, 1]$, $1/p + 1/q = 1$, $p > 1$.

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CITED LITERATURE

1. S. G. Kreĭn, Yu. I. Petunin, Scales of Banach spaces, UMN 21, no. 2 (128) (1966).
2. L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Semioordered Spaces*, Moscow-Leningrad, 1950.
3. A. P. Calderon, *Studia Math.*, 24, No. 2 (1964); A. P. Calderon, *Sbornik, Transl., Mathematics*, 9 (3) (1965).
4. G. Ya. Lozanovskii, *DAN*, 158, No. 3 (1964).
5. G. G. Lorentz, *Ann. Math.*, (2), 51, 37 (1950).

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