

SET-THEORETIC OPERATIONS

V. I. Amstislavskii

1966

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Abstract

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UDC 519.50

MATHEMATICS

V. I. AMSTISLAVSKII

SET-THEORETIC OPERATIONS

AND RECURSIVE HIERARCHIES

(Presented by Academician A. I. Mal'tsev on 24 XI 1965)

A general definition is given of a recursive hierarchy (with ω_1^* stages) in an arbitrary space. This definition may be regarded as an analogue of the definition of the classical hierarchies ⁽⁶⁾, which makes it possible to apply methods of the theory of operations on sets to establish a number of properties of recursive hierarchies. The hyperarithmetical ⁽⁹⁾ and ω_1 -stage $\mathfrak{S}$ -hierarchies ⁽⁸⁾ turn out to be equivalent in the sense specified in Theorems 5 and 6 to the effective (eff.) B - and, respectively, eff. C -hierarchies, **obtained in certain particular cases from our general definition***.

1. K -numberings. $N = \{0, 1, 2, \dots\}$; e, i, m, n are variables on N . Subsets of N are called **chains**, and sets of chains are called **bases**. A collection $\mathfrak{M}$ of arbitrary objects, supplied with some numbering α , is called a **family** and is denoted $\langle \mathfrak{M}, \alpha \rangle$, or simply α . Let $\langle \mathfrak{M}_1 \alpha_1 \rangle$ and $\langle \mathfrak{M}_2 \alpha_2 \rangle$ be two arbitrary families; let $P(x, y)$ be a predicate defined on $\mathfrak{M}_1 \times \mathfrak{M}_2$. We shall say: "for every $x (\in \mathfrak{M}_1)$ one can find $y (\in \mathfrak{M}_2)$ such that $P(x, y)$," meaning by this: "there exists a partial recursive function (p.r.f.) $f(n)$ such that if n is an α_1 -number of some $x \in \mathfrak{M}_1$, then $f(n)$ is defined and is equal to the α_2 -number of such a $y \in \mathfrak{M}_2$ that $P(x, y)$." The family of n -ary p.r.f.'s is assumed to be supplied with a Gödel numbering ⁽¹⁾; for $n = 1$, the p.r.f. with number e is denoted by $\langle e \rangle$; o is the one-place nowhere-defined p.r.f. The family of all recursively enumerable subsets of N is supplied with the numbering τ : $\forall e. \tau e =$ the domain of definition of the p.r.f. $\langle e \rangle$. For arbitrary families α and β , $\alpha \leq [e]\beta$ (and $\alpha \simeq [e]\beta$) means that the numbering α is reduced to the numbering β by means of the p.r.f. (recursive permutation) $\langle e \rangle$, i.e. $\alpha n = \beta \langle e \rangle n$; $\alpha \leq \beta \iff (\exists e)(\alpha \leq [e]\beta)$; and $\alpha \simeq \beta \iff (\exists e)(\alpha \simeq [e]\beta)$. If $\{\alpha_n\}_{n \in A}$ and $\{\beta_m\}_{m \in B}$ ($A, B \subseteq N$) are two sequences of families and for any $n \in A$, $m \in B$ one can find a number $f(n, m)$ such that $\alpha_n \leq [f(n, m)]\beta_m$, then we shall say that $\alpha_n \leq \beta_m$ uniformly in n, m . A simple numbering α will be called a K -numbering if $(\forall e_1, e_2)(\langle e_1 \rangle = \langle e_2 \rangle \Rightarrow \alpha e_1 = \alpha e_2)$. If m_0 is some number of the p.r.f. o , then αm_0 will be denoted by o_α . For example, the family τ is K -numbered. The

numbering $\langle e \rangle$ is effectively complete ⁽⁵⁾. Denote by e^* such a general recursive function (g.r.f.) that for every e , $\langle e^* \rangle$ is also a g.r.f. and for every n

$$\langle \langle e^* \rangle(n) \rangle = \begin{cases} \langle \langle e \rangle(n) \rangle, & \text{if } \langle e \rangle(n) \text{ is defined,} \\ o, & \text{otherwise;} \end{cases}$$

* ω_1 is the least nonconstructive ordinal number ⁽¹⁰⁾.

** This clarifies the connection between $\mathfrak{S}$ - and eff. C -hierarchies ⁽⁸⁾.

*** For the concepts and notation used below see ^(1,4) or ^(5,6).

then if α is a K -enumeration, then

$$\alpha \langle e^* \rangle(n) = \begin{cases} \alpha \langle e \rangle(n), & \text{if } \langle e \rangle(n) \text{ is defined,} \\ o_\alpha & \text{otherwise.} \end{cases} \quad (1)$$

2. Standard operators. Let E be an arbitrary set, called a space; let Ψ_M be a set-theoretic operation (s.t.o.) with base M . A function which assigns to each simply enumerated family α of subsets of the space E a family $\beta = H_M \alpha$, defined as follows: $\forall e. \beta e = \Psi_M \{ \alpha \langle e^* \rangle(n) \}$, will be called a **standard operator with base M** . From (1) it is clear that if α is a K -enumerated family, then $H_M \alpha$ is the family of all sets obtained as a result of the s.t.o. Ψ_M applied to all possible enumerated sequences of sets of the family α . A **composite standard operator with base** is a function which assigns to a sequence $\{ \alpha_n \}_{n \in N}$ of simply enumerated families the family $\beta = H_M \{ \alpha_n \}$, where $\forall e. \beta e = \Psi_M \{ \alpha_n \langle e^* \rangle(n) \}$. If all

$$\alpha_n = \alpha, \quad \text{then } H_M \{ \alpha_n \} = H_M \alpha.$$

In what follows, all K -enumerated families are families of subsets of the space E .

Theorem 1. *If all α_n , $n \in N$, are K -enumerated families, then $H_M \{ \alpha_n \}$ is also a K -enumerated family.*

Theorem 2. *If for each n the families β_n are simple, and α_n are K -enumerated, then for any general recursive function $\langle e \rangle$ such that $\alpha_n \leq [\langle e \rangle(n)] \beta_n$, one can find a number $h(e)$ such that*

$$H_M \{ \beta_n \} \leq [h(e)] H_M \{ \alpha_n \}.$$

Let a general recursive function $\sigma(m, n)$ map N^2 one-to-one onto N , and let $(M \mid L_i)$ be the base of such an s.t.o. that, for any sequence $\{ a_i \}_{i \in N}$ of families of sets,

$$\Psi_M\{\Psi_{L_i}\{a_{in}\}\} = \Psi_{M|L}\{\beta m\},$$

where $\beta\sigma(i, n) = a_{in}$.

Theorem 3. *There exists a number e_0 such that, for any K -enumerated family α and any bases $M, L_0, \dots, L_i, \dots$,*

$$H_M\{H_{L_i}\alpha\} \cong [e_0]H_{(M|L_i)}\alpha.$$

3. Definition of recursive hierarchies. Let $(O, <_0)$ be Kleene's system of notations for constructive ordinal numbers ⁽¹⁰⁾. Let $a' = 2^a$, if $a > 0$, and let the partial recursive function $[a]_i$ be such that if $a \in O$ and $a = 2^{(a)_0}$, then for every i , $[a]_i$ is defined,

$$[a]_i <_0 [a]_{i+1} <_0 a \quad \text{and} \quad \lim_i |[a]_i| = |a|.$$

Let $\mathcal{H}r(n)$ be such a general recursive predicate that

$$(n \in O \text{ and } \mathcal{H}r(n)) \iff (|n| \text{ is a limit nonzero ordinal number}).$$

We shall say that a base M **has property** (A) if $N \in M$ and all chains belonging to M are infinite. By M^c is denoted the base of the s.t.o. complementary to Ψ_M .

Basic definition. Let the base M or M^c have property (A) , and let λ be a K -enumerated family of subsets of the space E , with $\emptyset$ and E belonging to λ . The sequence $\{\lambda_a\}_{a \in O}$ of families of subsets of the space E , defined as follows:

- 1) $\lambda_1 = \lambda$;
- 2)

$$\lambda_{a'} = \begin{cases} H_M \lambda_a, & \text{if } \mathcal{H}r(a'), \\ H_{M^c} \lambda_a & \text{otherwise;} \end{cases}$$

- 3) if $a \neq 2^{(a)_0}$, then

$$\lambda_a = H_{M^c}\{\lambda_{[a]_i}\}$$

is called a **recursive hierarchy** (r.h.) with base M and initial family λ , or, more briefly, an $\langle M, \lambda \rangle$ -hierarchy.

It follows from Theorem 1 that each λ_a , $a \in O$, is a K -numbered family. By $\bar{\alpha}n$ we shall denote the family of complements of the sets of the family α : $\bar{\alpha}n = E \setminus \alpha n$. Theorems 4-6 are proved by effective induction over $(O, <_0)$ ⁽¹¹⁾.

Theorem 4. For any $a, b \in O$, $a <_0 b \Rightarrow \lambda_a \leq \lambda_b$ uniformly in ab ; if, moreover, $\bar{\lambda} \leq' H_M \lambda$, then $a <_0 b \Rightarrow \lambda_a \leq \lambda_b$ uniformly in a, b .

4. Relation with the hyperarithmetical and $\mathfrak{S}$ -hierarchies. Here $E = N$. Let, for $a \in O$, the predicates $H_a(x)$ and $\mathfrak{S}_a(x)$ be defined as in ^(8,9); put

$$h_a n = x(\exists y)T_1^{H_a}(n, x, y), \quad \mathfrak{h}_a n = x(\forall f)(\exists y)T_1^{\mathfrak{S}_{a'}f}(n, x, y).$$

Theorem 5. Let M be a base of the t.m.o. of intersection, and let $\{\tau_a\}_{a \in O}$ be an r.u. with base M and initial family τ . Then for every $a \in O$

$$\tau_a \simeq \begin{cases} h_{a'}, & \text{if } \mathfrak{H}r(a'), \\ \bar{h}_{a'}, & \text{otherwise} \end{cases}$$

uniformly in a .

Theorem 6. Let M be a complete base of the A -operation and let $\lambda = H_M ct$. Then the $\langle M, \lambda \rangle$ -hierarchy is such that, for $a \in O$,

$$\lambda_a \simeq \begin{cases} \mathfrak{h}_{a'}, & \text{if } \mathfrak{H}r(a'), \\ \bar{\mathfrak{h}}_{a'}, & \text{otherwise} \end{cases}$$

uniformly in a .

5. Properties of recursive hierarchies. We denote the totality of all chains by Y and shall regard it as a T_0 -space with the following topology ⁽⁷⁾: for any finite chain ξ' , the set $\sigma_{\xi'}$ of all chains containing ξ' as a subset is called **elementarily open**; an **open set** is any union of elementarily open sets. Let $\alpha = \{\alpha n\}$, $\alpha n \subseteq E$, $n \in N$; the function $F : E \rightarrow Y$, defined as follows:

$$\forall x \in E. F(x) = n \quad (x \in \alpha n),$$

will be called the **component function** corresponding to the sequence α . From the definition of a t.m.o. it is clear that $\Psi_M\{\alpha n\} = F^{-1}(M)$, where F is the component function corresponding to α . The class of continuous mappings of Y into itself coincides with the class of component functions corresponding to all possible sequences of open sets. An **effectively open set** will mean any enumerable union of elementarily open sets; denote by γ the numbering of the family of effectively open sets induced by the numbering τ . A component function $F : Y \rightarrow Y$ corresponding to some γ -enumerable sequence of effectively open sets will be called a **computable operation**. This definition is equivalent to the definition of a computable operation ⁽⁷⁾.

Theorem 7 (fixed point theorem). If the function $F : Y \rightarrow Y$ is continuous, then there exists a chain ξ_0 such that $F(\xi_0) = \xi_0$. If, moreover, F is a computable operation, then ξ_0 is a recursively enumerable subset of N .

Let $\langle \mathfrak{M}, \alpha \rangle$ be a family of bases; a base $L \in \mathfrak{M}$ will be called **universal** in the family $\langle \mathfrak{M}, \alpha \rangle$ if, for any base $M \in \mathfrak{M}$, one can find a computable operation F such that $M = F^{-1}(L)$.

Theorem 8. Let $\{\lambda_a\}_{a \in O}$ be an r.u. with base M and initial family λ in the space E , and let $\{\gamma_a\}_{a \in O}$ be an r.u. with the same base M and initial family γ of effectively open sets in Y . Then for every $a \in O$, in the family γ_a one can find a base L_a such that: 1) L_a is universal in γ_a ; 2) if $\gamma_a \leq H_M \gamma$, then for every $b <_0 a$, L_a does not belong to γ_b ; 3) $\lambda_a \simeq H_{L_a} \lambda$ uniformly in a .

Remark. 2) follows from (1) and Theorem 9; thus, the “non-emptiness” of the families of $\langle M, \gamma \rangle$ -hierarchies is established by means of the fixed point theorem instead of the traditional diagonal method.

A simply numbered family α is called **e.f.f. closed with respect to enumerable sums** if for every e one can find a number $s(e)$ such that

$$\bigcup_{n \in \tau e} \alpha n = \alpha s(e).$$

Analogously one defines **e.f.f. closedness with respect to pairwise intersections**.

Theorem 9 (an analogue of Kolmogorov’s “empty classes” theorem ⁽²⁾).

Let $\langle M, \lambda \rangle$ -hierarchy be such that λ is an e.f.f. closed with respect to enumerable sums and pairwise intersections, and in λ there exists an enumerable sequence of nonempty, pairwise disjoint sets. Then, for any $a \in O$, in the family λ_a one can find a set that belongs to no λ_b with $b <_O a$.

The hypotheses of Theorem 9 are satisfied, for example, when $E = N$ and $\lambda = \tau$, or E is Baire space and λ is a family of e.f.f. open sets ⁽³⁾.

We shall say that a sequence $\{\alpha_a\}_{a \in O}$ of families is **cofinal** in a sequence $\{\beta_b\}_{b \in O}$ if, for every $a \in O$, one can find a number $h(a)$ such that $\alpha_a \leq \beta_{h(a)}$ uniformly in a .

Theorem 10. *Consider two r.s. in a space E , having bases L and M , and one and the same initial family λ . If the base L belongs to one of the families of the $\langle M, \gamma \rangle$ -hierarchy in the space Y , then the $\langle L, \lambda \rangle$ -hierarchy is cofinal in the $\langle M, \lambda \rangle$ -hierarchy.*

Institute of Cybernetics
Academy of Sciences of the Azerbaijan SSR

Received
18 XI 1965

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