

# ON THE POLES AND THE RADIUS OF CIRCULAR UNIVALENCE OF THE MEROMORPHIC CLASS $\backslash(\backslash\Phi\backslash)$

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**Abstract**

**Full Text**

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**MATHEMATICS**

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# ON THE POLES AND THE RADIUS OF CIRCULAR UNIVALENCE OF THE MEROMORPHIC CLASS $\Phi$

(Presented by Academician M. A. Lavrent'ev, 27 XI 1965)

In <sup>(1)</sup> we considered one of the subclasses of the meromorphic class  $\Phi$ :

$$f(z) = \sum_{k=1}^n \sum_{s=1}^{m_k} \frac{A_{ks}}{(z - a_k)^s}, \quad (1)$$

$$a_k = \delta_k e^{i\varphi_k}, \quad \delta_k > 0, \quad m_k \geq 1 - \text{integers},$$

$$\arg A_{ks} - (s+1)(\pi + \arg a_k) \equiv \text{const} \equiv \varphi \pmod{2\pi} \quad (2)$$

for all  $k, s$  with  $A_{ks} \neq 0$ , and established the radius of a circle with center at  $z = 0$  such that every function of the subclass under consideration is univalent in it. In the present study we solve the problem completely.

**Theorem 1.** *Let  $R(\delta_k, m_k)$  be the greatest of the numbers  $R$  such that every function of the class  $\Phi$  is univalent in the circle  $|z| \leq R$ . Then*

$$R(\delta_k, m_k) = \min_k \delta_k \sin \frac{\pi}{2(m_k + 1)}. \quad (3)$$

The proof of Theorem 1 is based on two lemmas and two theorems.

**Lemma 1** (a generalization of the lemma from <sup>(1)</sup>). *Let  $p_1, p_2, \dots, p_n$ ,  $n \geq 1$ , be arbitrary real numbers, and suppose the powers  $(1 - z_k)^{p_k}$ ,  $k = 1, 2, \dots, n$ , have their principal value. Then the real part of the function*

$$P(z_1, \dots, z_n) = (1 - z_1)^{p_1} \dots (1 - z_n)^{p_n}$$

*is positive for arbitrary complex numbers  $z_1, \dots, z_n$  lying in the circle*

$$|z| < \sin \frac{\pi}{2p}, \quad p = \sum_{k=1}^n |p_k|$$

when  $p > 1$ , and in the circle  $|z| < 1$  when  $0 < p \leq 1$ . The numbers  $\sin(\pi/2p)$  for  $p > 1$  and 1 for  $p \leq 1$  cannot be replaced by larger ones.

**Proof.** From the inequalities  $|\arg(1 - z)| < \gamma$  for  $|z| < \sin \gamma$ ,  $0 < \gamma_0 \leq \pi/2$ , it follows that

$$|\arg P(z_1, \dots, z_n)| < \gamma \sum_{k=1}^n |p_k| = p\gamma.$$

Putting here  $\gamma = \gamma_0 = \pi/2p$  when  $p > 1$ , or  $\gamma = 1$  when  $0 < p \leq 1$ , we obtain the first assertion of the lemma. To prove the unimprovability of these bounds for  $p > 1$ , set, for  $p_k > 0$ ,  $z_k = \sin(\pi/2p)e^{-i\alpha}$ , where  $\alpha = (\pi/2p)(p-1)$ , and, for  $p_k < 0$ ,  $z_k = \sin(\pi/2p)e^{i\alpha}$ ; it is easy to verify that in this case  $\arg P(z_1, \dots, z_n) = \pi/2$ . If  $p \leq 1$  and at least one of the numbers  $z_k$  is equal to 1, then either  $P(z_1, \dots, z_n) = 0$ , or  $P$  has no meaning. Thus both circles in the lemma are the largest possible with the required properties.

**Lemma 2.** Let the function  $f(z)$  be holomorphic in the circle  $|z - z_0| < r_0$ , univalent in the closed circle  $|z - z_0| \leq c$  ( $c < r_0$ ), and let on its circumference  $|z - z_0| = c$  the derivative  $f'(z) \neq 0$ . Then there exists at least one  $c' > c$  such that  $f(z)$  is univalent in the closed circle  $|z - z_0| \leq c$ .

**Proof.** Otherwise, for any sequence  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  ( $\varepsilon_n > 0$ ), there will be found a sequence of pairs of points  $z_1^{(n)}, z_2^{(n)}$  such that  $z_1^{(n)} \neq z_2^{(n)}$ ,  $|z_j^{(n)} - z_0| \leq c + \varepsilon_n$  ( $j = 1, 2$ ), and  $f(z_2^{(n)}) = f(z_1^{(n)})$ .

Let  $z_j^{(n_k)}$  be a convergent subsequence,  $z_j^{(n_k)} \rightarrow z_j^*$ . Clearly  $z_j^*$  lies in the disk  $|z - z_0| \leq c$ , and by the continuity of  $f(z)$  we have  $f(z_1^*) = f(z_2^*)$ . Since  $f$  is univalent in the disk  $|z - z_0| \leq c$ , it follows that  $z_1^* = z_2^*$ . Consequently, in every neighborhood of  $z_1^*$  there lies a pair of points  $z_1^{(n_k)}$  and  $z_2^{(n_k)}$  such that  $z_1^{(n_k)} \neq z_2^{(n_k)}$  and  $f(z_1^{(n_k)}) = f(z_2^{(n_k)})$ . Under the assumptions made about the univalence of  $f$ , the point  $z_1^*$  cannot lie in the disk  $|z - z_0| < c$ . However, the case  $|z - z_0| = c$  is impossible, since  $f'(z_1^*) \neq 0$ , and therefore  $f(z)$  is univalent in a neighborhood of  $z_1^*$ . The lemma is proved.

**Theorem 2.** The disk  $|z| \leq R(\delta_k, m_k)$  (where  $R(\delta_k, m_k)$  is defined by equality (3)) is a domain of univalence for every function of the class  $\Phi$ .

**Proof.** For  $z_1 \neq z_2$  the equality holds

$$\frac{-e^{-i\varphi}(f(z_2) - f(z_1))}{z_2 - z_1} = \sum_{k=1}^n \sum_{s=1}^{m_k} \frac{s}{\delta_k^{s+1}} |A_{ks}| \int_0^1 \frac{dt}{(1 - z/a_k)^{s+1}}, \quad (4)$$

where  $z = (1-t)z_1 + tz_2$ , and it is assumed that the segment  $z_1z_2$  does not pass through a pole of  $f(z)$ . According to Lemma 1, the real parts of the integrand functions are positive when

$$\left| \frac{z}{a_k} \right| < \sin \frac{\pi}{2(m_k+1)} \leq \sin \frac{\pi}{2(s+1)}, \quad k = 1, \dots, n; \quad s = 1, \dots, m_k. \quad (5)$$

Consequently, for  $|z_1|, |z_2| < \min_k \delta_k \sin \pi/2(m_k+1)$  (recall that  $\delta_k = |a_k|$ ) the right-hand side of equality (4) is nonzero. Thus, for  $z_1 \neq z_2$  we have  $f(z_1) \neq f(z_2)$ , as was required to prove.

By the criterion given in Lemma 2, the disk  $|z| \leq R$  will be the largest disk in which  $f(z)$  is univalent if on its circumference there is at least one point  $z_0$  such that  $f'(z_0) = 0$ . In the following theorem all functions of the class  $\Phi$  with this property are described.

**Theorem 3.** *Let an arbitrary function of the class  $\Phi$  have a zero on the circumference of the disk of radius  $R$  with center at  $z = 0$ . Then  $f(z)$  has the form*

$$e^{-i\varphi} f(z) = \quad (6)$$

$$= \sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_{\sigma}} \frac{d_{k\sigma} \exp \left\{ i(m_{\sigma}+1) \left( \pi + \varphi_{11} + \frac{\pi \varepsilon_{11} m_1}{2(m_1+1)} - \frac{\pi \varepsilon_{k\sigma} m_{\sigma}}{2(m_{\sigma}+1)} \right) \right\}}{\left\{ z - R \operatorname{cosec} \frac{\pi}{2(m_{\sigma}+1)} \exp \left( i\varphi_{11} + \frac{i\pi \varepsilon_{11} m_1}{2(m_1+1)} - \frac{i\pi \varepsilon_{k\sigma} m_{\sigma}}{2(m_{\sigma}+1)} \right) \right\}^{m_{\sigma}}},$$

where

$$\sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_{\sigma}} \varepsilon_{k\sigma} m_{\sigma} d_{k\sigma} \left\{ \frac{1}{R} \lg \frac{\pi}{2(m_{\sigma}+1)} \right\}^{m_{\sigma}+1} = 0. \quad (7)$$

Only for these functions is the closed disk  $|z| \leq R$  the largest disk of univalence.

**Proof.** For the derivative of a function of the class  $\Phi$  we have

$$-e^{-i\varphi} f'(z) = \sum_{k=1}^n \sum_{s=1}^{m_k} \frac{s}{\delta_k^{s+1}} |A_{ks}| \left( 1 - \frac{z}{a_k} \right)^{-s-1}. \quad (8)$$

I. First of all, if for some  $z$  with  $|z| = R$  we have  $f'(z) = 0$ , then  $A_{ks} = 0$  for  $s < m_k$ ,  $k = 1, \dots, n$ . Indeed, from  $|z| \leq |a_k| \sin \frac{\pi}{2(m_k+1)} <$

$< \delta_k \sin \frac{\pi}{2(s+1)}$  and  $|z| \leq R$  it follows that

$\operatorname{Re}\{(1 - z_n/a_k)^{-s-1}\} > 0$ , i.e.  $f'(z) \neq 0$  for  $|z| = R$ , and  $A_{ks} \neq 0$  with  $s < m_k$ .

II. Further, if  $f'(z)$  vanishes on the circle  $|z| = R$ , then

$$\delta_k \sin \frac{\pi}{2(m_k + 1)} = R \quad (9)$$

for all  $k$  (otherwise, among the nonnegative quantities  $\operatorname{Re}(1 - z/a_k)^{-s-1}$  there would be a strictly positive one, and the equality  $f'(z) = 0$  would be impossible for  $|z| = R$ ).

III. From I and II we find the form of those functions of the class  $\Phi$  whose first derivative vanishes on the circle  $|z| = R$ :

$$f(z) = \sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_{\sigma}} \frac{A_{k\sigma}}{(z - a_{k\sigma})^{m_{\sigma}}}, \quad (10)$$

where  $a_{k\sigma} \neq a_{\sigma k}$  for  $k \neq \sigma$ ,  $a_{k\sigma} = \delta_{\sigma} e^{i\varphi_{k\sigma}}$  for  $k = 1, \dots, \nu_{\sigma}$ ;  $\delta_{\sigma} < \delta_a$  and  $m_{\sigma} < m_a$  for  $\sigma < a$ . The derivative of a function of the form (10) is equal to (we put  $A_{k\sigma} = d_{k\sigma} \exp(i\beta_{k\sigma})$ )

$$-e^{-i\varphi} f'(z) = \sum_{\sigma=1}^{\tau} \sum_{k=1}^{\nu_{\sigma}} m_{\sigma} d_{k\sigma} \left\{ \delta_{\sigma} \left( 1 - \frac{z}{a_k} \right) \right\}^{-m_{\sigma}-1}. \quad (11)$$

According to Lemma 1, from the equality  $f'(\zeta) = 0$ ,  $\zeta = Re^{i\varphi}$ , it follows that

$$\zeta = \delta_{\sigma} e^{i\varphi_{k\sigma}} \sin \frac{\pi}{2(m_{\sigma} + 1)} \exp \left( \frac{i\pi m_{\sigma} \varepsilon_{k\sigma}}{2(m_{\sigma} + 1)} \right), \quad k = 1, \dots, \nu_{\sigma}; \sigma = 1, \dots, \tau, \quad (12)$$

where  $\varepsilon_{k\sigma} = \pm 1$  must not have a constant sign. From (9), (11), and (12) we obtain (7). Further, from (12) we have

$$\varphi_{k\sigma} \equiv \varphi_{11} + \frac{\pi \varepsilon_{11} m_1}{2(m_2 + 1)} - \frac{\pi \varepsilon_{k\sigma} m_{\sigma}}{2(m_{\sigma} + 1)} \pmod{2\pi}. \quad (13)$$

Further,

$$\zeta = R \exp \left( i\varphi_{11} + \frac{i\varepsilon_{11} \pi m_1}{2(m_1 + 1)} \right) = \frac{1}{2} a_{11} \left( 1 - \exp \left( -\frac{i\pi \varepsilon_{11}}{m_1 + 1} \right) \right).$$

From the condition that a function of the form (10) belongs to the class  $\Phi$ , and from (13), we find

$$\beta_{k\sigma} \equiv \varphi + (m_{\sigma} + 1) \left( \pi + \varphi_{11} + \frac{\pi \varepsilon_{11} m_1}{2(m_1 + 1)} - \frac{\pi \varepsilon_{k\sigma} m_{\sigma}}{2(m_{\sigma} + 1)} \right) \pmod{2\pi}. \quad (14)$$

Substituting (9), (13), and (14) into (10), we obtain (6), as was required to prove.

If we put  $m_1 = \max_{\sigma} m_{\sigma}$ , then from (6) it immediately follows that  $\tau = 1$ ,  $k = 1, \dots, \nu_1$ , and (5) becomes  $\sum_{k=1}^{\nu_1} \varepsilon_{k1} d_{k1} = 0$ . Hence  $\sum'_k d_{k1} = \sum''_k d_{k1} = d$ , where  $\sum'_k$ ,  $\sum''_k$  denote sums over those  $k$  for which  $\varepsilon_{k1} = +1$ , respectively  $\varepsilon_{k1} = -1$ . Consequently, (6) turns into

$$e^{-i\varphi} f(z) = \tag{6'}$$

$$= \frac{d \exp(i(m_1 + 1)(\pi + \varphi_{11}))}{\left(z - R \operatorname{cosec} \frac{\pi}{2(m_1 + 1)} e^{i\varphi_{11}}\right)^{m_1}} + \frac{d \exp\{i(m_1 + 1)(\pi + \varphi_{11}) + i\pi \varepsilon_{11} m_1\}}{\left(z + R \operatorname{cosec} \frac{\pi}{2(m_1 + 1)} e^{i(\varphi_{11} - \frac{\pi \varepsilon_{11}}{m_1 + 1})}\right)^{m_1}},$$

i.e. we have obtained those, among the functions considered in (1), for which the disk  $|z| \leq R$  is the largest disk of univalence.

Theorem 1 follows from Theorem 2 and Theorem 3.

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## CITED LITERATURE

1. Pavel Todorov, DAN, 162, No. 2 (1965).

*Note: Figure translations are in progress. See original paper for figures.*

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