

ON QUASICONFORMAL MAPPINGS IN SPACE

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Abstract

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MATHEMATICS

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ON QUASICONFORMAL MAPPINGS IN SPACE

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In the present note Q -quasiconformal mappings ⁽¹⁾ in three-dimensional Euclidean space are considered.

1. Let D be a ring domain bounded by the sphere $|X| = 1$ and the segment $[0, r]$, and let $M(D)$ be its modulus. Then ⁽⁴⁾

$$\ln \frac{1}{r} \leq M(D) \leq \ln \frac{\lambda}{r}, \quad 4 \leq \lambda \leq 12.4 \dots$$

Since $M(D) = 2\sqrt{\pi} M(\Sigma)$, where Σ is the family of all possible surfaces σ separating the boundary components of D , it follows that

$$\frac{1}{2\sqrt{\pi}} \ln \frac{1}{r} \leq M(\Sigma) \leq \frac{1}{2\sqrt{\pi}} \ln \frac{\lambda}{r}.$$

These estimates (with the constant $\lambda = 18$) were obtained earlier by another method by B. V. Shabat.

Hence, as also in ⁽¹⁾, it follows:

Theorem 1. Let $Y = f(X)$, $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$, $f(0) = 0$, be an arbitrary Q -quasiconformal mapping of the ball $|X| < 1$ onto itself.

Then at every point of the ball the estimates

$$(|X|/\lambda)^Q \leq |f(X)| \leq \lambda |X|^{1/Q}, \quad 4 \leq \lambda \leq 12.4 \dots \quad (1)$$

hold.

We shall suppose that an extremal Q -quasiconformal mapping of the ball $|X| < 1$ onto itself, giving the maximum distortion $\rho = \rho(r)$, carries at least one plane section passing through $[0, r]$ into a plane section passing through $[0, \rho]$. Then the distortion under the Q -quasiconformal mapping $Y = f(X)$, $f(0) = 0$, of the

ball $|X| < 1$ onto itself will not exceed the distortion under the Q -quasiconformal mapping $w = \varphi(z)$, $\varphi(0) = 0$, of the disk $|z| < 1$ onto itself:

$$|\varphi(z)| \leq \rho,$$

where ρ is determined from the relation

$$\frac{K'(\rho)}{K(\rho)} = \frac{1}{Q} \frac{K'(r)}{K(r)},$$

in which $K(r)$ and $K'(r)$ are, respectively, the complete elliptic integrals of the first kind with moduli $k = r$ and $k' = \sqrt{1 - r^2}$ (²), or, as a consequence of this,

$$4^{1-Q}|z|^Q \leq |\varphi(z)| \leq 4^{1-1/Q}|z|^{1/Q}.$$

On the other hand, the author has constructed an example of a Q -quasiconformal mapping of the ball $|X| < 1$ onto itself for which

$$\lim_{r \rightarrow 0} \frac{\rho}{r^{1/Q}} = 4^{1-1/Q}.$$

The example is based on a mapping obtained by rotating a plane extremal mapping of the disk onto itself. This shows that the constant λ in (1) cannot be replaced by any smaller one than $4^{1-1/Q}$.

Therefore, under the stated assumption, we have

Theorem 2. Let $Y = f(X)$, $f(0) = 0$, be an arbitrary Q -quasiconformal mapping of the ball $|X| < 1$ onto itself.

Then at every point of the ball the estimates hold

$$4^{1-Q}|X|^Q \leq |f(X)| \leq 4^{1-1/Q}|X|^{1/Q},$$

and the constant $4^{1-1/Q}$ cannot be improved.

2. Let $Q(D)$ be the lower bound of the numbers Q for which there exist Q -quasiconformal mappings of the domain D onto the ball $|Y| < 1$; if such mappings do not exist, we shall put $Q(D) = \infty$. Following Gehring (⁵), we shall call $Q(D)$ the **coefficient of quasiconformality** of the domain D , and any $Q(D)$ -quasiconformal mapping of the domain D onto the ball $|Y| < 1$ an **extremal mapping**.

Lemma. Let the function $w = \varphi(z)$ map Q -quasiconformally a domain G , whose boundary contains at least one angular point P with angle $\pi\alpha$, $0 \leq \alpha \leq 2$, onto a domain G^* with smooth boundary, and suppose there exists a neighborhood U of the point P in which

$$|\Delta w/\Delta z| \geq k \quad \text{for } 0 \leq \alpha \leq 1; \quad |\Delta w/\Delta z| \leq 1/k \quad \text{for } 1 < \alpha \leq 2, \quad (2)$$

where k is some constant.

Then $Q \geq \max(\alpha, 1/\alpha)$.

Let $0 \leq \alpha \leq 1$, and let, for sufficiently small $\varepsilon > 0$, Γ_ε be the family of all possible curves joining the boundary components of the annulus $\varepsilon^2 < |z - P| < \varepsilon$ in U . Then

$$M(\Gamma_\varepsilon) = \frac{1}{\pi\alpha} \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right). \quad (3)$$

On the other hand, under any mapping with $Q < 1/\alpha$, in view of (2) and (3), Theorem 5, the images of the arcs of the circles $|z - P| = \varepsilon$, $|z - P| = \varepsilon^2$ will lie in the annuli $k\varepsilon \leq |\varphi(z) - P^*| \leq K\varepsilon$, $k\varepsilon^2 \leq |\varphi(z) - P^*| \leq K\varepsilon^2$, respectively; $K = K(Q, k)$.

Hence

$$M(\Gamma_\varepsilon^*) = \frac{1}{\pi} \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right) \quad (4)$$

and from (3), (4)

$$M(\Gamma_\varepsilon^*) = \alpha M(\Gamma_\varepsilon) + o\left(\ln \frac{1}{\varepsilon}\right),$$

i.e. $Q \geq 1/\alpha$.

This contradiction proves the assertion of the lemma. The proof in the case $1 < \alpha \leq 2$ is analogous.

We shall say that a spatial domain D contains on its boundary a smooth edge AB with angle $\pi\alpha$, $0 \leq \alpha \leq 2$, if at every point $P \in AB$ the cross-section normal to AB has angle $\pi\alpha$.

Theorem 3. Let the function $Y = f(X)$ map Q -quasiconformally a domain D , containing on its boundary a smooth edge AB with angle $\pi\alpha$, $0 \leq \alpha \leq 2$, onto the ball $|Y| < 1$, and suppose that at every point $P \in AB$ the cross-section normal to AB is transformed into a smooth surface.

Then $Q \geq \max(\alpha, 1/\alpha)$.

The mapping $f(X)$ can be extended to a homeomorphism of $D \cup AB$ onto $\{|Y| < 1\} \cup f(AB)$ ⁽⁶⁾. Therefore there is a point $P \in AB$ at which, in the cross-section normal to AB , (2) is satisfied, and application of the lemma gives the assertion of the theorem.

It follows from this that the assertion of Theorem 3 remains valid if the mapping $f(X)$ satisfies the smoothness condition only at one of those points $P \in AB$ at which (2) is fulfilled on the normal sections. In the class of such mappings we have

Corollary. *If a domain D contains on its boundary a smooth edge with angle $\pi\alpha$, $0 \leq \alpha \leq 2$, then $Q(D) \geq \max(\alpha, 1/\alpha)$, and for $\alpha = 0$, $Q(D) = \infty$.*

Let (r, φ, x_3) , (ρ, θ, y_3) be cylindrical coordinates in the spaces X and Y , respectively, and let D be the dihedral angle $0 < \varphi < \pi\alpha$, $0 < \alpha \leq 2$. Since the mapping

$$\rho = r, \quad \theta = \varphi/\alpha, \quad y_3 = x_3 \quad (5)$$

transforms D into a half-space and for it $Q = \max(\alpha, 1/\alpha)$, it is extremal by Theorem 3.

Thus, for a dihedral angle $\pi\alpha$,

$$Q(D) = \max(\alpha, 1/\alpha).$$

The extremal mapping (5) is not unique; the mappings

$$\rho = kr, \quad \theta = \varphi/\alpha, \quad y_3 = \varphi(x_3),$$

where $\varphi(x_3) \in C^1$, $\varphi'(x_3)$ lies between k and k/α , and k is an arbitrary constant, will also be such.

If, in particular, D is a quarter-space ($\alpha = 1/2$) or the space with the half-plane $\{x_1 > 0, x_2 = 0\}$ removed ($\alpha = 2$), then $Q(D) = 2$.

However, if D is one-eighth of space (an octant), then the double unfolding is not an extremal mapping. Using orthogonal projection, we see that in this case

$$Q(D) \leq 2 + \sqrt{3} \approx 3.72.$$

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