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O. A. Oleinik

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Abstract

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MATHEMATICS

O. A. OLEINIK

ON THE SYSTEM OF BOUNDARY-LAYER EQUATIONS FOR AN UNSTEADY FLOW OF AN INCOMPRESSIBLE FLUID

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1. Consider the system of boundary-layer equations for a nonstationary two-dimensional flow of a viscous incompressible fluid

$$u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \quad u_x + v_y = 0 \quad (1)$$

in the domain $D\{0 \leq t < t_0, 0 \leq x < x_0, 0 \leq y < \infty\}$, where $t_0 \leq \infty$, $x_0 \leq \infty$, with the conditions

$$u|_{t=0} = u_0(x, y), \quad u|_{y=0} = 0, \quad v|_{y=0} = v_0(t, x), \quad u|_{x=0} = u_1(t, y), \quad (2)$$

$$\lim_{y \rightarrow \infty} u(t, x, y) = U(t, x). \quad (3)$$

The functions $p(t, x)$ and $U(t, x)$ are connected by Bernoulli's law $U_t + UU_x = -p_x$ (see (1)). We shall show that, under certain smoothness conditions on p , U , u_0 , v_0 , u_1 and natural compatibility conditions for these functions on the axes t , x , and y , there always exists a solution u , v of this system in the domain D for any $x_0 \leq \infty$, if t_0 is sufficiently small, and also in the domain D for any $t_0 \leq \infty$, if x_0 is sufficiently small. This means that in the indicated regions of variation of t , x , y there are no separation points of the boundary layer. The system of equations of the stationary boundary layer was studied in the work (2).

We assume that $u_0 > 0$ and $u_1 > 0$ for $y > 0$, $u_{0y} > 0$, $u_{1y} > 0$ for $y \geq 0$, and $U > 0$ for all t and x from D .

To prove the existence of a solution of problem (1)–(3) in the indicated domains, we pass to new independent variables

$$\tau = t, \quad \xi = x, \quad \eta = u(t, x, y). \quad (4)$$

Then, for the new unknown function $w = u_y$, we obtain the equation

$$L(w) \equiv \nu w^2 w_{\eta\eta} - w_\tau - \eta w_\xi + p_x w_\eta = 0 \quad (5)$$

in the domain $\Omega\{0 \leq \tau < t_0, 0 \leq \xi < x_0, 0 \leq \eta < U(\tau, \xi)\}$, with boundary conditions

$$w|_{\tau=0} = u_{0y} \equiv w_0(\xi, \eta), \quad w|_{\xi=0} = u_{1y} \equiv w_1(\xi, \eta), \quad w|_{\eta=U(\tau, \xi)} = 0, \quad (6)$$

$$l(w) \equiv \nu w w_\eta - v_0 w - p_x = 0 \quad \text{for } \eta = 0. \quad (7)$$

We assume that p , U , w_0 , v_0 , w_1 are sufficiently smooth functions.

2. We obtain the solution of problem (5)–(7) as the limit of a sequence of functions w^n as $n \rightarrow \infty$, satisfying the equation

$$L_n(w^n) \equiv \nu(w^{n-1})^2 w_{\eta\eta}^n - w_\tau^n - \eta w_\xi^n + p_x w_\eta^n = 0 \quad (8)$$

in the domain Ω , the conditions (6), and the condition

$$l_n(w^n) \equiv \nu w^{n-1} w_\eta^n - v_0 w^{n-1} - p_x = 0 \quad \text{for } \eta = 0. \quad (9)$$

As w^0 we choose a sufficiently smooth function satisfying conditions (6) and such that $w^0 > 0$ for $\eta < U(\tau, \xi)$.

Assume first that there exist in Ω solutions w^n of the linear problem (8), (6), (9), which have continuous derivatives up to third order in $\bar{\Omega}$, and show that w^n converge as $n \rightarrow \infty$ to a solution of problem (5)–(7), and then prove the existence of w^n .

Lemma 1. There exists a constant τ_0 such that, for all n and $\tau \leq \tau_0$, in the domain Ω the inequalities $h_1 \leq w^n \leq H_1$ hold, where $h_1(\tau, \xi, \eta)$, continuous in $\bar{\Omega}$, satisfies $h_1(\tau, \xi, \eta) > 0$ for $\eta < U(\tau, \xi)$, $\tau \leq \tau_0$, and H_1 is a bounded function.

Lemma 2. There exists a constant ξ_0 such that, for all n and $\xi \leq \xi_0$, in the domain Ω the inequalities $h_2 \leq w^n \leq H_2$ hold, where $h_2(\tau, \xi, \eta)$, continuous, satisfies $h_2(\tau, \xi, \eta) > 0$ for $\eta < U(\tau, \xi)$, $\xi \leq \xi_0$; H_2 is a function bounded in Ω .

We prove these lemmas by induction, with a suitable choice of the functions h_i and H_i ($i = 1, 2$), using the maximum principle for equations of the form (8).

To estimate the first- and second-order derivatives of w^n , we consider in Ω the function

$$V_1^n = W_\tau^2 + W_\xi^2 + W_\eta(W_\eta - 2H) + K_1\eta + K_2,$$

where $W \equiv e^{an}w^n$, $H \equiv v/v + p_x/vw^{n-1}e^{an} + aw^n e^{an}\psi(\eta)$, and the function

$$V_2^n = W_{\xi\xi}^2 + W_{\tau\xi}^2 + W_{\tau\tau}^2 + W_{\xi\eta}(W_{\xi\eta} - 2H_\xi) + \\ + W_{\tau\eta}(W_{\tau\eta} - 2H_\tau) + \chi^2(\eta)W_{\eta\eta} + K_3\eta + K_4,$$

where $\chi(\eta)$ and $\psi(\eta)$ are smooth functions, $\chi(0) = 0$, $\chi(\eta) = 1$ for $\eta > \delta$, $\psi(\eta) = 1$ for $\eta < \delta/2$, and $\psi(\eta) = 0$ for $\eta > \delta$, $\delta > 0$ being some small number; a, K_1, K_2, K_3, K_4 are some positive constants. The functions p_x and v_0 in H are extended for $\eta > 0$ so that they are equal to zero for $\eta \geq \delta$.

We prove, using Lemmas 1 and 2 and the equations satisfied by V_1^n and V_2^n , that the constants a, K_1, K_2, K_3, K_4 can be chosen so that there exist numbers M_1, M_2 , and τ_1 having the following properties: $M_1 > V_1^0$, $M_2 > V_2^0$, and, if $M_1 > V_1^m$ and $M_2 > V_2^m$ for $m \leq n-1$ and $\tau \leq \tau_1$, then $M_1 > V_1^n$ and $M_2 > V_2^n$ for $\tau \leq \tau_1$. From the boundedness of V_1^n and V_2^n with respect to n there follows the boundedness of the first- and second-order derivatives of w^n for $\tau \leq \tau_1$. Their boundedness with respect to n in Ω for $\xi \leq \xi_1$ is proved analogously, where $\xi_1 \leq \xi_0$, and ξ_1 is determined by the data of problem (1)–(3). The uniform convergence of w^n is proved using the maximum principle for the equation satisfied by the difference $w^n - w^{n-1}$.

Theorem 1. The functions w^n converge uniformly as $n \rightarrow \infty$, together with their first-order derivatives, to a function w , which is a solution of problem (5)–(7) in the domain Ω , if $t_0 \leq \tau_1$ or if $x_0 \leq \xi_1$.

In what follows we shall consider the domains Ω under the condition that $t_0 \leq \tau_1$ or $x_0 \leq \xi_1$.

3. We indicate one method for constructing the functions w^n . (We note that analogous methods were applied to the study of linear equations of the form (8) in [3].) Below we shall specify a boundary-value problem for an elliptic equation in a certain special domain, whose solutions $w^{\varepsilon n}$ converge uniformly to w^n as $\varepsilon \rightarrow 0$. The corresponding boundary-value problem for a parabolic equation can be constructed analogously.

Let G be an infinitely differentiable bounded domain in the ξ, η -plane such that the cylinder $G \times [0, t_0]$ contains Ω , and the boundary σ of the domain G contains the segment $[-2\delta, x_0 + 2\delta]$ of the ξ -axis, where $\delta > 0$ is some small number. We shall suppose that, in some neighborhood of the point A of intersection of σ with the line $\xi = 0$, the boundary σ lies on the straight line $\eta = \text{const}$. Consider a

simply connected infinitely differentiable domain Q with boundary S , coinciding with the cylinder $G \times [-1, t_0 + 1]$ for $-1 \leq \tau \leq t_0 + 1$ and contained in the cylinder $G \times [-2, t_0 + 2]$. The points of the domain Q for which $\tau \geq 0$, $\xi \geq 0$ or $\tau \geq t_0$, we denote by Ω_1 . We extend smoothly the coefficient p_x in equation (8) and v_0 and p_x in the boundary condition (9) for all values of ξ and τ .

Let $\bar{w}$ be a smooth function defined in $Q - \Omega_1$ and satisfying-

satisfying the conditions: $\bar{w}|_{\tau=0} = w_0$, $\bar{w}|_{\xi=0} = w_1$, $L(\bar{w}) = O(\xi^4)$ for $\xi \leq 0$ and $\tau \geq 0$ in a neighborhood of the plane $\xi = 0$ for $\eta \leq U(\tau, 0)$, $L(\bar{w}) = O(\tau^4)$ for $\xi \geq 0$, $\tau \leq 0$ in a neighborhood of the plane $\tau = 0$ for $\eta \leq U(0, \xi)$, $l(\bar{w}) = O(\xi^4)$ on S in a neighborhood of the segment $[0, t_0]$ of the τ -axis and $l(\bar{w}) = O(\tau^4)$ in a neighborhood of the segment $[0, x_0]$ of the ξ -axis; moreover $L(\bar{w})$ has, in $Q - \Omega_1$, bounded derivatives up to order 4 inclusive. Such a function $\bar{w}$ can be constructed if w_0 , w_1 , v_0 , p_x are sufficiently smooth functions and, in addition, w_0 and w_1 satisfy the compatibility conditions on the τ, ξ, η axes in accordance with equation (5) and the boundary conditions (6) and (7). For example, the function $\bar{w}$ can be defined as follows:

$$\bar{w} = w_1 + \xi \left. \frac{\partial w}{\partial \xi} \right|_{\xi=0} + \dots + \frac{\xi^k}{k!} \left. \frac{\partial^k w}{\partial \xi^k} \right|_{\xi=0}, \quad k \geq 4,$$

for $\xi \leq 0$ and $\tau \geq 0$, where the derivatives of w with respect to ξ at $\xi = 0$ are determined from equation (5) and from the equations obtained from it by differentiation with respect to ξ , under the condition that $w = w_1$ for $\xi = 0$. For $\tau \leq 0$, $\bar{w}$ is defined by the formula

$$\bar{w} = \bar{w}|_{\tau=0} + \tau \left. \frac{\partial w}{\partial \tau} \right|_{\tau=0} + \dots + \frac{\tau^k}{k!} \left. \frac{\partial^k w}{\partial \tau^k} \right|_{\tau=0}, \quad k \geq 4,$$

where the derivatives of w with respect to τ at $\tau = 0$ are equal respectively to the derivatives of $\bar{w}$ at $\tau = 0$ for $\xi \leq 0$, and for $\xi \geq 0$ they are determined from equation (5) and from the equations obtained from it by differentiation with respect to τ . As w^0 in (8) and (9) we choose $\bar{w}$. Let $\sigma_\delta = \sigma - q_\delta$, where q_δ is the segment $[-\delta, x_0 + \delta]$ of the ξ -axis, $S_\delta = \sigma_\delta \times [-1, t_0 + 1]$. In the domain Q consider the operator

$$\begin{aligned} L^\varepsilon(w) \equiv & \varepsilon(w_{\tau\tau} + w_{\xi\xi} + w_{\eta\eta}) + a_1 w_{\tau\tau} + a_2 w_{\xi\xi} + a_3 w_{\eta\eta} \\ & + (w^{n-1})_\varepsilon^2 w_{\eta\eta} - w_\tau - \eta w_\xi + (p_x)_\varepsilon w_\eta - 2(\varepsilon + a_1)w. \end{aligned}$$

Here $\varepsilon > 0$, the functions a_1, a_2, a_3 are positive for $\tau < -1/2$ and for $\tau > t_0$; a_3 is also positive in the δ -neighborhood S_δ , and a_2 is positive everywhere in this neighborhood except for its points lying on the plane $\xi = 0$. At the remaining points of Q the functions a_1, a_2, a_3 are equal to zero. By $(f)_\varepsilon$ we shall denote the

averaging with radius ε of the function f by means of an infinitely differentiable kernel.

Consider in the domain Q the boundary-value problem for the elliptic equation

$$L^\varepsilon(w) = (F)_\varepsilon \quad (10)$$

with boundary condition

$$\partial w / \partial \bar{n} = (\Phi)_\varepsilon, \quad (11)$$

where $\bar{n}$ is the direction of the interior normal to S . The function F occurring in (10) is defined in Q in the following way:

$$F = L(\bar{w}) - 2a_1\bar{w} + a_1\bar{w}_{\tau\tau} + a_2\bar{w}_{\xi\xi} + a_3\bar{w}_{\eta\eta}$$

in $Q - \Omega_1$, $F = 0$ in Ω , and F is an arbitrary smooth extension of this function (with bounded derivatives of 4th order) in the remaining part of the domain Q . Put $\Phi = v_0/\nu + p_x/\nu w^{n-1}$ on

$$S_0 = \{0 \leq \xi \leq x_0, \quad 0 \leq \tau \leq t_0\}$$

and $\Phi = \partial \bar{w} / \partial \bar{n}$ on $S \cap Q - \Omega_1$. On the remaining part of S the function Φ occurring in (11) is an arbitrary smooth extension of the indicated function. It is obvious that, by virtue of the properties of $\bar{w}$, the functions F and Φ have bounded derivatives of 4th order. The solution $w^{\varepsilon n}$ of problem (10), (11) exists and is infinitely differentiable in the closed domain Q (see, for example, (4)). It can be shown that the $w^{\varepsilon n}$ are bounded and have in Q derivatives bounded uniformly in ε up to order 4 inclusive.

Theorem 2. *The solutions $w^{\varepsilon n}$ of problem (10), (11) in Q , as $\varepsilon \rightarrow 0$, converge uniformly in Ω , together with derivatives up to order 3 inclusive, to the function w^n , which is the solution of problem (8), (6), (9).*

The fact that the limiting function for $w^{\varepsilon n}$ as $\varepsilon \rightarrow 0$ satisfies

conditions (6), follows from the fact that this limit is equal to $\bar{w}$ in $Q - \Omega_1$ and satisfies the equation $w_\tau + \eta w_\xi - p_x w_\eta = 0$ on the surface $\eta = U(\tau, \xi)$.

From Theorems 1 and 2 there follows the existence of a solution of problem (5)–(7) in the indicated domains Ω . Passing by formulas (4) to the independent variables t, x, y and expressing $u(t, x, y)$ in terms of w , we obtain a solution of problem (1)–(3).

The uniqueness of the solution of problem (5)–(7), and consequently also of (1)–(3), is easily obtained by using the maximum principle. In an analogous way one can prove its continuous dependence on w_0 and w_1 . A proof of uniqueness of the solution of these problems is also contained in the work (5).

It can be proved that the constructed solution $u(t, x, y)$ in problem (1)–(3) is stable as $t \rightarrow \infty$ in the sense that if the prescribed functions $p(t, x)$, $U(t, x)$, $v_0(t, x)$ tend uniformly in x , as $t \rightarrow \infty$, respectively to certain functions $p(x)$, $U(x)$, $v_0(x)$, and $u_1(t, y)$, for sufficiently large t , does not depend on t , then as $t \rightarrow \infty$ the function $u(t, x, y)$ tends to the longitudinal component of the velocity $u(t, x)$ of the stationary boundary layer corresponding to the limiting functions $p(x)$, $U(x)$, $v_0(x)$, $u_1(y)$. A detailed proof of this assertion is given in (6).

Moscow State University
named after M. V. Lomonosov

Institute for Problems in Mechanics
Academy of Sciences of the USSR

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CITED LITERATURE

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