

DESCRIPTION OF COMPLETELY IRREDUCIBLE REPRESENTATIONS OF A SEMISIMPLE COMPLEX LIE GROUP

D. P. Zhelobenko, M. A. Naimark

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.84663>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.512.2

MATHEMATICS

D. P. ZHELOBENKO, M. A. NAIMARK

DESCRIPTION OF COMPLETELY IRREDUCIBLE REPRESENTATIONS OF A SEMISIMPLE COMPLEX LIE GROUP

(Presented by Academician A. N. Kolmogorov, 12 I 1966)

Since the time when tensor algebra and representation theory came into existence, many mathematicians have been concerned with the question of describing all irreducible representations of one group G or another, i.e., with finding all irreducible solutions of the functional equation

$$F(g_1 g_2) = F(g_1) F(g_2)$$

in some class of linear operators. Thus, for a complex semisimple connected group G , É. Cartan and H. Weyl gave a complete classification of all irreducible representations in finite-dimensional spaces. The analogous problem for infinite-dimensional representations has a more complicated history and encounters much greater difficulties of both an analytic and a topological nature.

Meanwhile, since the appearance of the first works of I. M. Gelfand and M. A. Naimark on infinite-dimensional representations, the answer to the indicated problem had in fact been anticipated. It was natural to suppose that all the sought solutions are contained in some way in the series of “elementary” representations $e(a)$ (for a description see ⁽¹⁾), which is connected by analytic continuation with the “principal series” of Gelfand and Naimark. Harish-Chandra ⁽²⁾ developed a deep infinitesimal apparatus for solving this problem; Harish-Chandra’s ideas were subsequently developed by R. Godement ⁽³⁾ and others. Another approach, based on the Plancherel formula, was proposed by one of the authors of this article ⁽⁴⁾ as applied to the class of unitary representations of classical groups. Finally, F. A. Berezin ⁽⁵⁾, using his results on Laplace operators on the group G , gave a positive solution of the problem for representations in Banach spaces.

In the present note, the result of F. A. Berezin is carried over to an arbitrary locally convex complete linear space. Moreover, this result is refined, since in the space of an elementary representation, if it is reducible, the desired irreducible subspace is indicated. In doing so, essentially the same method is used as in

(⁴), but instead of the Plancherel formula, which is now no longer sufficient, a theorem of the Paley-Wiener type, proved in (⁶), is used.

Throughout this article we shall denote by the symbol G a semisimple complex connected Lie group, and by the symbol T_g its continuous representation in a locally convex complete space E .

1. The first difficulty in working with nonunitary representations consists in the reasonable introduction of the basic notions of representation theory: irreducibility and equivalence. In this note we shall use the following definitions:

Definition 1. Let $S(E)$ be the algebra of all weakly continuous endomorphisms of the space E . The representation T_g is called **completely**

irreducible*, if the weakly closed linear envelope of the operators T_g coincides with the whole algebra $S(E)$.

Definition 2. Two representations, T_g and S_g , in the spaces E and F are called **weakly equivalent** if there exists a linear operator $A : E \rightarrow F$, mapping one-to-one a linear invariant submanifold $E_0 \subset E$, everywhere dense in E , onto a linear submanifold $F_0 \subset F$ possessing the same properties, and moreover

$$AT_g = S_gA$$

for every element g of the group G , and if there exists a formally adjoint operator $A^* : F' \rightarrow E'$, possessing all the same properties with respect to the adjoint representations of the group G .

Proposition 1. *If the representation T_g is completely irreducible, then: 1) it is topologically irreducible; 2) its contragredient representation is also completely irreducible; 3) an analogue of Schur's lemma holds: every linear weakly closed operator commuting with all operators T_g is a scalar multiple of the identity.*

Proposition 2. *For the class of unitary representations of the group G in Hilbert spaces, the notion of complete irreducibility is equivalent to the notion of topological irreducibility, and the notion of weak equivalence is equivalent to the notion of unitary equivalence.*

We note that weak equivalence does not, generally speaking, possess the property of transitivity**.

2. Along with the representation T_g , we shall consider the corresponding representation

$$T_x = \int x(g)T_g dg$$

of the group algebra X , where X is the totality of all infinitely differentiable finite functions $x(g)$ on the group G with the usual topology. The operator

function T_x is continuous on X with respect to this topology. The definitions of irreducibility and equivalence carry over verbatim to the representation T_x , and the fulfillment of any one of these properties is equivalent to the fulfillment of the same property for the original representation T_g of the group G . Further, let $\mathfrak{U}$ be a maximal compact subgroup in the group G . Any continuous function $c(u)$ on the group $\mathfrak{U}$ may be regarded as a measure on G , concentrated on the subset $\mathfrak{U}$. We note that the space X is invariant with respect to convolutions cx , xc with such measures c , and the formula

$$T_c = \int c(u) T_u du$$

allows one to extend the representation T_x to the measures c . In particular, let k be some highest weight of the subgroup $\mathfrak{U}$, and let $e = e_k$ be one of the minimal projectors with highest weight k , generated by compact shifts $u \in \mathfrak{U}$ on the group G^{***} ; then the operator

$$p = T_{e_k}$$

is a projection operator in the space E . Using such operators, it is easy to carry out the decomposition of the representation T_g into representations,

* See (1). This definition generalizes the one given by R. Godement for Banach spaces.

** However, in the class of completely irreducible representations of a semisimple Lie group, transitivity does hold. Moreover, in this case weak equivalence is equivalent to infinitesimal equivalence.

*** The number of such projectors is finite and is equal to the dimension of the irreducible representation $\delta(k)$ of the group $\mathfrak{U}$ with highest weight k .

multiple irreducibles: $\delta(k_0), \delta(k_1), \delta(k_2), \dots$, where the symbol k_0 denotes the minimal one among the highest weights in the space E .

Let us pass to the main construction.

3. Criterion of weak equivalence. Fix one of the minimal projectors $e = e_k$ with highest weight k , and consider in the algebra X the subalgebra

$$Y = eXe.$$

Let P be the corresponding projector in the space E ; then it is evident that the representation of the subalgebra Y is localized in the subspace PE . Introduce the notation A_y = the restriction of T_y to PE , and call the representation

A_y associated with the original representation of the algebra X . Then the following holds.*

Lemma 1. *If the representation of the algebra X is completely irreducible, then the associated representation of the algebra Y is also completely irreducible. If two representations of the algebra X are completely irreducible and their associated representations are nonzero and weakly equivalent, then the representations of the algebra X are also weakly equivalent.*

If the representation $\delta(k)$ is contained in the space E , then the projector P , and along with it the associated representation of the subalgebra Y , are different from the identically zero one. Consequently, the enumeration of all completely irreducible representations of the algebra X is reduced to the analogous problem for the subalgebra Y .

4. We can now obtain the main result. Let $\{e(a)\}$ be the system of all elementary representations of the group G . Recall ⁽¹⁾ that each such representation is either completely irreducible, or contains a finite number of completely irreducible components. Among the latter, a special role is played by the components occurring with minimal highest weight k_0 . We shall agree to call such components **minimal**.** We shall prove that the following holds.

Theorem. *Every completely irreducible representation of the group G is weakly equivalent to one of the elementary representations $e(a)$ or to its minimal component.*

Proof. From the Pelley-Wiener theorem ⁽⁶⁾ for the algebra X it follows that the algebra $\tilde{Y}$, dual to Y , contains an ideal I of the form

$$I = \sum_{l < k} \overline{X^{kl}} \overline{X^{lk}},$$

where X^{lk} is the space dual to $e_l X e_k$, and the index l runs through the system of all highest weights subordinate to the highest weight k . Moreover, the quotient algebra F is an algebra, with respect to multiplication, consisting of entire analytic functions $f(\rho) = f(\rho_1, \rho_2, \dots, \rho_r)$ (r is the rank of the group G) with certain growth conditions and a symmetry condition of the form $f(s\rho) = f(\rho)$, where s runs through the subgroup of the Weyl group defined as the stationary subgroup of the highest weight k . Suppose now that $k = k_0$ is the minimal highest weight in the space E ; then the ideal I is mapped to zero, and hence it follows that

Lemma 2. *If the representation T_x is completely irreducible and the subalgebra Y is constructed from the minimal highest weight k , then its associated representation is one-dimensional.*

Indeed, the associated representation A_y is a completely irreducible representation of the commutative quotient algebra F ; it remains only to—

* For the Lorentz group see ⁽⁷⁾; the proof in the general case is analogous.

** Minimal components can always be realized ⁽¹⁾ in the invariant subspace $e(a)$.

apply Schur's lemma. As a result,

$$A_y = \mu(f),$$

where $\mu(f)$ is a multiplicative continuous functional on the algebra F . But every such functional, as is easy to verify, has the form $f(\rho_0)$, where ρ_0 is some fixed point. It remains now to verify that there exists an elementary representation, or its minimal component, which possesses the same multiplicative functional. The theorem is proved.

5. In particular, if we are interested in the question of classifying unitary representations, then from the general theorem we obtain

Corollary. *Every unitary irreducible representation of the group G is unitarily equivalent either to one of the elementary representations $e(\alpha)$, or to its minimal component.*

To make this result more precise, it suffices to determine which of the elementary representations of the group G admit the introduction of an invariant scalar product either in the whole space or at least in its minimal component.* Thus the question of the classification of all unitary irreducible representations of the group G may be regarded as completely solved. At the same time, if the representation is nonunitary, there remains the possibility (within the framework of weak equivalence) of varying the topology in the representation space.

Peoples' Friendship University
named after P. Lumumba

Received
14 XII 1965

REFERENCES

- ¹ D. P. Zhelobenko, DAN, **170**, No. 5 (1966).
- ² Harich-Chandra, Trans. Am. Math. Soc., **70**, 1 (1951); review by Zh. Diksmye in *Collection of translations. Mathematics*, **6**, 5 (1962).
- ³ R. Godement, Trans. Am. Math. Soc., **73**, 3 (1954).
- ⁴ M. A. Naimark, *Mathematical Collection*, **35**, No. 2, 317 (1954); **37**, No. 1, 121 (1955).
- ⁵ F. A. Berezin, *Transactions of the Moscow Mathematical Society*, **6**, 371 (1957); **12**, 453 (1963).
- ⁶ D. P. Zhelobenko, DAN, **170**, No. 6 (1966).
- ⁷ M. A. Naimark, *Linear Representations of the Lorentz Group*, Moscow, 1958.

* If positive definiteness of the scalar product is not required, then one can show that this condition is equivalent to the identities

$$-\bar{p} = sq, \quad -\bar{q} = sp,$$

where (p, q) is the signature of the representation under study and s is some transformation from the Weyl group.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.