

ON SOME STABILITY CONDITIONS CONNECTED WITH THE STUDY OF RESONANCES

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Abstract

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MATHEMATICS

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ON SOME STABILITY CONDITIONS CONNECTED WITH THE STUDY OF RESONANCES

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§ 1. Statement of the problem. Consider a system of equations describing fast and slow motions (systems of this type often occur in applications, see ^(1,2,4-7)) of the form: $\dot{x} = \varepsilon X(x, y, \varepsilon) = \varepsilon X_1(x, y) + \varepsilon^2 \dots$, $\dot{y} = Y(x, y, \varepsilon) = Y_0(x) + \varepsilon Y_1(x, y) + \varepsilon^2 \dots$, where $\varepsilon > 0$ is a small parameter, $x = \{x_1, \dots, x_n\}$ is the set of slowly varying variables, $y = \{y_1, \dots, y_m\}$ is the set of "fast" variables, $X_j = \{X_j^{(1)}, \dots, X_j^{(n)}\}$, $Y_k = \{Y_k^{(1)}, \dots, Y_k^{(m)}\}$ ($j = 1, 2, \dots$; $k = 0, 1, 2, \dots$). Suppose that for $\varepsilon = 0$ the system has an equilibrium point (x_0, y_0) (without restricting generality, one may put $x_0 = y_0 = 0$).

For $\varepsilon = 0$ the system has the form: $\dot{y} = Y_0(x)$, $x = \text{const}$. We shall regard the original system for $\varepsilon \neq 0$ as perturbed and pose the problem of the stability of the solution $x = y = 0$ with respect to perturbations of the initial values and of the right-hand sides. It is clear that if the equations $\dot{y} = Y_0(x)$, $x = \text{const}$ ($Y_0(0) = 0$) are taken as the unperturbed system, then the solution $x = y = 0$ of these equations will not be stable under constantly acting perturbations, since this solution is not asymptotically stable (because of the presence of the family of solutions $x = \text{const}$). Therefore, in the unperturbed system one should retain terms of order ε . Thus we arrive at the following problem. As the unperturbed system we take the equations $\dot{x} = \varepsilon X_1(x, y)$, $\dot{y} = Y_0(x) + \varepsilon Y_1(x, y)$ and assume that $X_1(0, 0) = Y_0(0) = 0$. Suppose that for sufficiently small values $\varepsilon \neq 0$ this system has an equilibrium point of the form $x(\varepsilon), y(\varepsilon)$, located near the origin $(0, 0)$ and admitting the expansion $x(\varepsilon) = \varepsilon\alpha + \varepsilon^2 \dots$, $y(\varepsilon) = \varepsilon\beta + \varepsilon^2 \dots$. The original equations will be regarded as perturbed. Assuming that the right-hand sides of the equations are sufficiently smooth in all arguments, we can now reduce the perturbed system to the form $\dot{u} = A(\varepsilon)u + F(u, \varepsilon)$. Here $u = \{u_1, \dots, u_{n+m}\} = \{x_1 - \varepsilon\alpha_1, \dots, x_n - \varepsilon\alpha_n, y_1, \dots, y_m\}$, $F = \{f_1, \dots, f_n, \varphi_1, \dots, \varphi_m\}$, and $A(\varepsilon)$ is a matrix of order $n + m$ of the form

$$A(\varepsilon) = \begin{pmatrix} \varepsilon a_{11} & \dots & \varepsilon a_{1n} & \varepsilon b_{11} & \dots & \varepsilon b_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon a_{n1} & \dots & \varepsilon a_{nn} & \varepsilon b_{n1} & \dots & \varepsilon b_{nm} \\ c_{11} + \varepsilon d_{11} & \dots & c_{1n} + \varepsilon d_{1n} & \varepsilon e_{11} & \dots & \varepsilon e_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1} + \varepsilon d_{m1} & \dots & c_{mn} + \varepsilon d_{mn} & \varepsilon e_{m1} & \dots & \varepsilon e_{mm} \end{pmatrix}, \quad (1)$$

where $a_{ik} = (\partial X_1^{(i)} / \partial x_k)_0$, $b_{jl} = (\partial X_1^{(j)} / \partial y_l)_0$, $c_{pq} = (\partial Y_0^{(p)} / \partial x_q)_0$, $d_{rs} =$

$$= \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial^2 Y_0^{(r)}}{\partial x_i \partial x_s} \right)_0 \alpha_i + \left(\frac{\partial Y_1^{(r)}}{\partial x_s} \right)_0, \quad e_{\alpha\beta} = (\partial Y_1^{(\alpha)} / \partial y_\beta)_0$$

(the notations $(\dots)_0$ indicate here that the corresponding expressions are taken at $x = 0$,

$y = 0$). If the right-hand sides of the original equations are sufficiently smooth, then, as is not difficult to verify,

$$\begin{aligned} f_i &= O(\varepsilon^2 + \varepsilon^2 \|u\| + \varepsilon \|u\|^2), \quad i = 1, \dots, n; \\ \varphi_k &= O(\varepsilon^2 + \varepsilon^2 \|u\| + \|u\|^2), \quad k = 1, \dots, m. \end{aligned}$$

These considerations lead to the formulation of the following general problem: take a system of equations of the form

$$\dot{z} = A(\varepsilon)z + \Phi(\varepsilon, z, t), \quad \varepsilon > 0, \quad (2)$$

where $A(\varepsilon)$ is the real matrix defined by formula (1), $z = \{z_1, \dots, z_{n+m}\}$, $\Phi = \{\Phi_1, \dots, \Phi_{n+m}\}$, and the functions Φ_i satisfy the conditions

$$\begin{aligned} \Phi_i &= O(\varepsilon^2 + \varepsilon^2 \|z\| + \varepsilon \|z\|^2), \quad i = 1, \dots, n; \\ \Phi_j &= O(\varepsilon^2 + \varepsilon^2 \|z\| + \|z\|^2), \quad j = 1, \dots, m. \end{aligned} \quad (3)$$

It is required to establish conditions under which the trivial solution $z = 0$ of the first-approximation system $\dot{z} = A(\varepsilon)z$ will be stable with respect to perturbations of the form (3) for all sufficiently small values $\varepsilon > 0$.

We note that this question cannot be resolved on the basis of the known general theorems on stability under continuously acting perturbations (see, for example, (3)), since the usual conditions $\operatorname{Re} \lambda_j(\varepsilon) < 0$, $j = 1, \dots, n + m$ ($\lambda_j(\varepsilon)$ are the eigenvalues of the matrix $A(\varepsilon)$), do not guarantee stability. Indeed, it is not hard to see that as $\varepsilon \rightarrow 0$, $\lambda_1(\varepsilon), \dots, \lambda_{n+m}(\varepsilon) \rightarrow 0$; hence perturbations of the form (3) may fail to be "sufficiently small." Examples can be given in which the trivial solution of a system of type (2) turns out to be unstable even under the

condition $\operatorname{Re} \lambda_j(\varepsilon) < 0$. Consequently, for stability, some additional conditions must be imposed on the matrix $A(\varepsilon)$, stronger than the requirements of the Hurwitz criterion. It also turns out that, in the case of stability, the perturbed solutions of system (2) are close to the rest point, generally speaking, only on a time interval of order $t \sim 1/\sqrt{\varepsilon}$.

§ 2. Main results. Sufficient algebraic conditions for stability have been obtained for the matrix (1); their essence is that they ensure the relations $\operatorname{Re} \lambda(\varepsilon) \leq -l\varepsilon$, where l is some positive constant. In the general case these conditions are rather cumbersome, and therefore we shall present them here for the special case $n \geq m = 1$.

Let: a) $S = \sum_{i=1}^n a_{ii} + e_{11} < 0$; b) $-k^2 = \sum_{i=1}^n b_{1i}c_{1i} < 0$; c) all roots of the equation $\operatorname{Det} B(\rho) = 0$, where

$$B(\rho) = \begin{pmatrix} a_{11} - \rho \dots a_{1n} & b_{11} \\ \cdot & \cdot \\ a_{n1} \dots a_{nn} - \rho & b_{n1} \\ c_{11} \dots c_{1n} & 0 \end{pmatrix}$$

are distinct and have negative real parts;

d)

$$\sum_{i_1 < i_2 < \dots < i_{n-2} \leq n} \Delta_{i_1, \dots, i_{n-2}} - k^2 S > 0,$$

where $\Delta_{i_1, \dots, i_{n-2}}$ ($i_k = 1, \dots, n$) are the diagonal minors of order three of the determinant of the matrix $B(0)$ (for $n = 1$ conditions c) and d) may be dropped, and for $n > 1$ condition a) is not needed).

It is not difficult to verify that, for sufficiently small ε , requirements a)–d) ensure, in particular, the fulfillment of the conditions of the Hurwitz criterion and, moreover,

Moreover, if conditions a)–d) are satisfied, then for all eigenvalues $\lambda(\varepsilon)$ of the matrix (1) the estimate $\operatorname{Re} \lambda(\varepsilon) \leq -l\varepsilon$ ($l = \operatorname{const} > 0$) is valid. If conditions a)–d) are satisfied, the trivial solution of system (2) (for $m = 1$) is stable in the following sense: for arbitrary $T > 0$ and $\gamma_1 > 0$ one can specify such $\varepsilon_0 > 0$ and $\gamma_2 > 0$ that, for all $0 < \varepsilon \leq \varepsilon_0$ and $t_0 \leq t \leq t_0 + T$, every solution of system (2) satisfying at the initial instant the condition $\|z(t_0)\| \leq \gamma_2$ and defined for all $t \in [t_0, t_0 + T]$ (uniqueness of the solution is not assumed), for all values $t \in [t_0, t_0 + T]$ satisfies the inequality $\|z(t)\| \leq \gamma_1$. This assertion can be refined; namely, it turns out that, generally speaking, $T \sim 1/\sqrt{\varepsilon}$, $\gamma_1 \sim \sqrt{\varepsilon}$, and $\gamma_2 \sim \varepsilon$: for arbitrary $\delta > 0$ there exist such $C_1, C_2, \varepsilon_0 > 0$ that, for $0 < \varepsilon \leq \varepsilon_0$, every solution of system (2) satisfying the condition $\|z(t_0)\| \leq \delta\varepsilon$ satisfies, for $t \in [t_0, t_0 + C_1/\sqrt{\varepsilon}]$, the inequality $\|z(t)\| \leq C_2\sqrt{\varepsilon}$.

In the general case, for an arbitrary number of fast and slow variables, and also in the presence of multiple characteristic roots, the stability conditions are formulated in analogous terms.

§ 3. Applications to resonance problems. In the theory of nonlinear oscillations one encounters systems of equations of the form:

$$\dot{q} = \varepsilon Q(q, \varphi, \varepsilon), \quad \dot{\varphi} = \omega(q) + \varepsilon \Phi(q, \varphi, \varepsilon), \quad (4)$$

where $q = \{q_1, \dots, q_r\}$, $\varphi = \{\varphi_1, \dots, \varphi_s\}$, $\omega = \{\omega_1, \dots, \omega_s\}$, and the right-hand sides are periodic in the variables $\varphi_1, \dots, \varphi_s$ ^(1,2,4-10). In the study of stationary resonant regimes, which may arise under an integer relation among the frequencies $\omega_1, \dots, \omega_s$, system (4) can, under certain conditions, be reduced to a system of type (2). Therefore the results of § 2 can be used in investigating the stability of stationary resonant oscillatory and rotational regimes.

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- ¹ N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Moscow, 1963. ² Yu. A. Mitropolsky, *Problems of the Asymptotic Theory of Nonstationary Oscillations*, Moscow, 1964. ³ I. G. Malkin, *Theory of Stability of Motion*, 1952. ⁴ N. N. Moiseev, *Zhurn. vychislit. matem. i matem. fiz.*, 3, No. 1 (1963). ⁵ F. L. Chernousko, *Zhurn. vychislit. matem. i matem. fiz.*, 3, No. 1 (1963). ⁶ V. M. Volosov, *UMN*, 17, issue 6 (108) (1962). ⁷ V. M. Volosov, *Zhurn. vychislit. matem. i matem. fiz.*, 3, No. 1 (1963). ⁸ V. M. Volosov, B. I. Morgunov, *DAN*, 153, No. 3 (1963). ⁹ V. M. Volosov, B. I. Morgunov, *DAN*, 156, No. 1 (1964). ¹⁰ V. M. Volosov, B. I. Morgunov, *DAN*, 161, No. 6 (1965).

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