

ON A METHOD FOR INVESTIGATING NONLINEAR SINGULAR EQUATIONS

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Abstract

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MATHEMATICS

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ON A METHOD FOR INVESTIGATING NON-LINEAR SINGULAR EQUATIONS

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In the present note a new method is proposed for investigating the equation

$$u(x) = \lambda \int_a^b \frac{f[x, s, u(s)]}{s - x} ds \quad (1)$$

in the Guseinov class $H_{\alpha, \beta, \delta}^K$ (1).

We first give several functional inequalities for functions from the Hölder class H, δ and $H_{\alpha, \beta, \delta}^K$ ($0 < \alpha + \delta, \beta + \delta, \alpha, \beta < 1$), which were obtained by one of the authors of the present note.

1°. Let $u(x) \in H_{\alpha, \beta, \delta}^K$. Then $W(x) = u(x)\psi(x)$ belongs to the class $H_{l_1, K, \delta}^0$, where $\psi(x) = (x - a)^{\alpha + \delta}(b - x)^{\beta + \delta}$, and $H_{l_1, K, \delta}^0$ is the Hölder class of functions that vanish at the endpoints of the interval (a, b) ,

$$l_1 = \left(\frac{5}{4}\right)^{\alpha + \delta} + 4^{-\alpha} \left(\frac{5}{4}\right)^{\beta + \delta} (b - a)^{\delta} + 4^{-\beta} (b - a)^{\delta}.$$

2°. If $u(x) \in H_{K, \delta}^0$, then $W(x) = u(x)(x - a)^{-\alpha - \delta}(b - x)^{-\beta - \delta}$ belongs to the class $H_{\alpha, \beta, \delta}^{l_2 K}$; l_2 is a constant independent of K .

3°. If $u(x) \in H_{K, \delta}$, then

$$\max_{a \leq x \leq b} |u(x)| \leq l K^{1/(1 + \delta p)} \left\{ \int_a^b |u(x)|^p dx \right\}^{\delta/(1 + \delta p)}, \quad p > 0;$$

$$l = \begin{cases} \frac{(b - a)^{p\delta^2/(1 + \delta p)}}{2^\delta} - \frac{\sqrt[p]{2}}{(b - a)^{\delta/(1 + \delta p)}}, & \text{if } b - a < \frac{2^{(1 + \delta p)/\delta p}}{(\delta p)^{1/\delta}} = \delta_0, \\ (\delta p)^{1/(1 + \delta p)} + \left(\frac{1}{\delta p}\right)^{\delta p/(1 + \delta p)}, & \text{if } b - a \geq \delta_0. \end{cases}$$

From 1° and 3° it follows:

4°. If $u(x) \in H_{\alpha, \beta, \delta}^K$, then

$$\sup_{a < x < b} \{|u(x)|\psi(x)\} \leq l(l_1 K)^{1/(1+\delta)} \left\{ \int_a^b \rho(x) |u(x)|^p dx \right\}^{\delta/(1+\delta p)},$$

where $\rho(x) = [\psi(x)]^p$.

Since $\rho(x) \leq (b-a)^{\bar{\gamma}} \rho_1(x)$, $\rho_1(x) = (x-a)^{\gamma} (b-x)^{\gamma'}$, $\bar{\gamma} = (\alpha+\delta)p + (\beta+\delta)p - \gamma - \gamma'$, $\varphi - 1 < \gamma \leq \alpha p + \delta p$, $\beta p - 1 < \gamma' \leq \beta p + \delta p$, from 4° we obtain:

5°. If $u(x) \in H_{\alpha, \beta, \delta}^K$, then

$$\sup_{a < x < b} \{|u(x)|\psi(x)\} \leq l_3 l(l, K)^{1/(1+\delta p)} \left\{ \int_a^b \rho_1(x) |u(x)|^p dx \right\}^{\delta/(1+\delta p)},$$

$$l_3 = (b-a)^{\gamma \delta / (1+\delta p)}.$$

6°. If $u(x) \in H_{K, \delta}$, then

$$\sup_{a \leq x, y \leq b} \frac{|u(x) - u(y)|}{|x - y|^{\delta'}} \leq 2^{1-\delta'/\delta} K^{\delta'/\delta} \{\max |u(x)|\}^{(\delta-\delta')/\delta}, \quad 0 < \delta' < \delta.$$

From 1° and 6° it follows

7°. For $u(x) \in H_{\alpha, \beta, \delta}^K$ one has

$$\begin{aligned} \sup_{a < x, y < b} \left\{ \frac{|u(x)\psi(x) - u(y)\psi(y)|}{|x - y|^{\delta'}} \right\} &\leq \\ &\leq 2^{(\delta-\delta')/\delta} (l, K)^{\delta'/\delta} \left\{ \sup_{a < x < b} |u(x)|\psi(x) \right\}^{(\delta-\delta')/\delta}. \end{aligned}$$

8°. If $u(x) \in H_{\alpha, \beta, \delta}^K$, then

$$\begin{aligned} \sup_{\substack{a < x, y < b \\ |y-x| \leq \sigma(x)}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\delta'}} (x-a)^{\alpha+\delta} (b-y)^{\beta+\delta} \right\} &\leq \\ &\leq l_4 K^{\delta'/\delta} \left\{ \sup_{a < x < b} |u(x)|\psi(x) \right\}^{(\delta-\delta')/\delta}, \end{aligned}$$

$$l_4 = \{(4/3)^{\alpha+\delta} + (5/4)^{\beta+\delta}\}^{(\delta-\delta')/\delta}, \quad \min \left\{ \frac{x-a}{4}, \frac{b-x}{4} \right\} = \sigma(x).$$

From inequalities 5° and 8° it follows that

$$\|u\|_1 \leq \tilde{f}_1 \left(\|u\|_{L_p(\rho_1)} \right), \quad (*)$$

where

$$\tilde{f}_1(x) = l_3 l(l, K)^{1/(1+\delta p)} x^{\delta p/(1+\delta p)} + l_4 \{l_3 l(l, K)^{1/(1+\delta p)}\}^{(\delta-\delta')/\delta} x^{p(\delta-\delta')/(1+\delta p)},$$

$$\begin{aligned} \|u\|_1 = & \sup_{a < x < b} \{|u(x)|\psi(x)\} + \\ & + \sup_{\substack{a < x < b \\ |\Delta x| < \sigma(x)}} \left\{ \frac{|u(x+\Delta x) - u(x)|}{|\Delta x|^{\delta'}} (x-a)^{\alpha+\delta} (b-x-\Delta x)^{\beta+\delta} \right\}, \end{aligned}$$

$$\|u\|_{L_p(\rho_1)} = \left\{ \int_a^b \rho_1(x) |u(x)|^p dx \right\}^{1/p}.$$

From inequalities 5° and 7° it follows that

$$\|u\|_2 \leq f_2 \left(\|u\|_{L_p(\rho_1)} \right), \quad (**)$$

where

$$\begin{aligned} f_1(x) = & l_3 l(l, K)^{1/(1+\delta p)} x^{\delta p/(1+\delta p)} + 2^{(\delta-\delta')/\delta} (l, K)^{\delta'/\delta} \times \\ & \times \{l_3 l(l, K)^{1/(1+\delta p)}\}^{(\delta-\delta')/\delta} x^{p(\delta-\delta')/(1+\delta p)}, \end{aligned}$$

$$\begin{aligned} \|u\|_2 = & \sup_{a < x < b} \{|u(x)|\psi(x)\} + \\ & + \sup_{\substack{a < x < b \\ |\Delta x| < \sigma(x)}} \left\{ \frac{|u(x+\Delta x)\psi(x+\Delta x) - u(x)\psi(x)|}{|\Delta x|^{\delta'}} \right\}. \end{aligned}$$

Theorem 1. Let $f(x, s, u)$ be defined for $a < s$, $s < b$, $-\infty < u < +\infty$ and satisfy the conditions

$$|f(x + \Delta x, s + \Delta s, u + \Delta u) - f(x, s, u)| \leq \frac{A\{|\Delta x|^{\delta'} + |\Delta s|^{\delta}\}}{(s - a)^{\alpha + \delta}(b - s - \Delta s)^{\beta + \delta}} + B|\Delta u|, \quad (2)$$

$$\int_a^b \frac{f(x, s, 0)}{s - x} ds \in H_{\alpha, \beta, \delta}^{K_0}, \quad 0 < \delta < \delta_1 \leq 1. \quad (3)$$

Suppose, further, that $F(x, s, u) = f(x, s, u) - f(s, s, u)$ satisfies the condition

$$|F(x, s, u) - F(x, s, v)| \leq g(x, s)|u - v|, \quad (4)$$

where $g(x, s)$ is a nonnegative function such that the integral

$$R_1 = \left\{ \int_a^b \rho_1(x) \left[\int_a^b \rho_1^{-q/p}(s) \left| \frac{g(x, s)}{s - x} \right|^q ds \right]^{1/p} \right\} \quad (5)$$

converges for some $p > 1$, $1/p + 1/q = 1$, $\gamma < 1$, $\gamma' < 1$. Then the operator

$$Mu = \lambda \int_a^b \frac{f[x, s, u(s)]}{s - x} ds$$

for small λ maps $H_{\alpha, \beta, \delta}^K$ into $H_{\alpha, \beta, \delta}^K$ and is a contraction operator in the space $H_{\alpha, \beta, \delta}^K$ with metric

$$\rho_2(u, v) = \|u - v\|_{L_p(\rho_1)}$$

provided

$$|\lambda| < \lambda_0 = \min\{K/K', 1/R_2\}, \quad K' = K_0 + C_0(2A + KB),$$

$$R_2 = C_1 B + R_1, \quad C_0 \text{ and } C_1 \text{ are certain constants.}$$

If one takes into account inequality (*), then we have proved

Theorem 2. If $f(x, s, u)$ satisfies the conditions of Theorem 1 and $|\lambda| < \lambda_0$, then equation (1) has a unique solution $u_0^*(x) \in H_{\alpha, \beta, \delta}^{\alpha K}$, and this solution can be found by the method of successive Picard approximations

$$u_n(x) = \lambda \int_a^b \frac{f[x, s, u_{n-1}(s)]}{s - x} ds, \quad u_0(x) \in H_{\alpha, \beta, \delta}^K.$$

The successive approximations converge in the sense of the metric

$$\rho_3(u, v) = \|u - v\|_2,$$

and

$$\rho_3(u_n, u_0^*) \leq f_1 \left\{ \frac{R_3^n}{1 - R_3} \rho_2(Mu_0, u_0) \right\}, \quad (6)$$

where $R_3 = |\lambda|R_2 < 1$.

If one takes into account that

$$\rho_2(Mu_0, u_0) \leq KR_4, \quad R_4 = 2 \int_a^b \tilde{\rho}_1(x) dx, \quad \tilde{\rho}_1(x) = (x-a)^{\gamma-\alpha p} (b-x)^{\gamma'-\beta p},$$

then, substituting its expression in place of $f_1(x)$, we obtain

$$\rho_3(u_n, u_0^*) \leq K \{ \tilde{l}_4 R_3^{n\delta p/(1+\delta p)} + l_5 R_3^{np(\delta-\delta')/(1+\delta p)} \}, \quad (6')$$

where

$$\begin{aligned} \tilde{l}_4 &= l_3 l_1 \left(\frac{R_4}{1 - R_3} \right)^{\delta p/(1+\delta p)}, \\ l_5 &= 2^{(\delta-\delta')/\delta} l_1^{(1+\delta'p)/(1+\delta p)} l_3^{(\delta-\delta')/\delta} \left(\frac{R_4}{1 - R_3} \right)^{p(\delta-\delta')/(1+\delta p)}. \end{aligned}$$

Inequalities of type (6') can also be obtained in the sense of the metric

$$\rho_4(u, v) = \|u - v\|_1.$$

It should be noted that inequality (6') for nonlinear singular equations has been obtained here for the first time.

On the basis of the inequalities given above, a theorem is proved which characterizes the dependence of the solution of equation (1) on the parameter λ .

Theorem 3. Suppose that $f(x, s, u)$ satisfies all the conditions of Theorem 2. Suppose, further, that $u(x, \lambda_1)$, $u(x, \lambda_2)$ are solutions of equation (1), corresponding to the parameters λ_1, λ_2 ($|\lambda_1| < \lambda_0$, $|\lambda_2| < \lambda_0$). Then

$$\sup_{a < x < b} \{|u(x, \lambda_1) - u(x, \lambda_2)|\psi(x)\} \leq R_6 |\lambda_1 - \lambda_2|^{\delta p / (1 + \delta p)},$$

$$R_6 = l(2l, K)^{1/(1+\delta p)} \left(\frac{R_5}{1 - R_2 |\lambda_2|} \right)^{\delta p / (1 + \delta p)}, \quad (7)$$

$$R_5^p = \int_a^b \rho_1(x) \left\{ \int_a^b \frac{f[x, s, u(s, \lambda_1)]}{s - x} ds \right\}^p dx.$$

If in inequality (7) we set $p = 1/(1 - \delta)$, taking $\gamma = \alpha p - 1 + \varepsilon$, $\gamma' = \beta p - 1 + \varepsilon_1$, $0 < \varepsilon, \varepsilon_1 < 1$, we obtain

$$\sup_{a < x < b} \{|u(x, \lambda_1) - u(x, \lambda_2)|\psi(x)\} \leq R_6 |\lambda_1 - \lambda_2|^\delta$$

under the condition that integral (5) converges.

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CITED LITERATURE

1. A. I. Guseinov, *Izv. AN SSSR, ser. matem.*, **12**, 193 (1948).

Note: Figure translations are in progress. See original paper for figures.

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