

ASYMPTOTIC FORMULAS

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Abstract

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MATHEMATICS

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ASYMPTOTIC FORMULAS

FOR A CERTAIN CLASS OF TRIGONOMETRIC SUMS

(Presented by Academician I. M. Vinogradov on 22 X 1965)

In this paper an asymptotic formula is obtained for a certain class of trigonometric sums, making it possible to judge the behavior of the modulus of such sums as the interval of summation changes.

We shall use the following notation: n, m, t, P, a are integers, $n \geq 20$; r is a real number, $1 \leq r \leq 0.1n$; $t \geq n$; p is a prime number, $(a, p) = 1$; $P \gg 1$;

$$\delta_n(x) = \begin{cases} 1, & \text{if } x \equiv 0 \pmod{n}, \\ 0, & \text{if } x \not\equiv 0 \pmod{n}. \end{cases}$$

Theorem 1. Let $q = p^t$, $P = q^{1/r}$. If

$$P = a_0 p^s + a_1 p^{s-1} + \dots + a_{s-1} p + a_s, \quad 1 \leq a_0 \leq p-1, \quad 0 \leq a_\nu \leq p-1, \quad \nu \geq 1,$$

is the p -adic expansion of the number P , then the asymptotic formula

$$S = \sum_{x=1}^P \exp 2\pi i \frac{ax^n}{q} = A a_0 p^{s-\alpha} + a_0 p^{s-\alpha+\beta} + O(P^{1-\gamma/r^2}),$$

holds, where the quantities A, α, β are determined by the equalities

$$A = \delta_n(t-1) \sum_{x=1}^{p-1} \exp 2\pi i \frac{ax^n}{p}, \quad \alpha = \left[\frac{t}{n} \right] + 1, \quad \beta = \delta_n(t).$$

The formula has a particularly simple form when $P = p^s$ and $t \equiv 0 \pmod{n}$. In this case we obtain:

$$a_0 = 1, \quad a_\nu = 0, \quad \nu = 1, 2, \dots, s; \quad \delta_n(t-1) = 0, \quad A = 0, \quad \delta_n(t) = 1;$$

$$P^r = p^{sr} = p^t, \quad s = t/r; \quad t/ns = r/n;$$

$$S = P^{1-r/n} + O(P^{1-\gamma/r^2}).$$

For the proof of the theorem we shall need the following lemma.

Lemma. Let $q = p^m$, $m \geq n + 1$, $(a_1, p) = \dots = (a_{n+1}, p) = 1$;

$$P^r = q, \quad 1 \leq r \leq 0.5n; \quad f(x) = a_1x + a_2px^2 + \dots + a_{n+1}p^n x^{n+1}.$$

Consider the trigonometric sum

$$S_L = \sum_{x=1}^P \exp 2\pi i \frac{f(x)}{q}.$$

Then the estimate

$$|S_L| \leq CP^{1-\gamma_1/r^2},$$

holds, where C and $\gamma_1 > 0$ are absolute constants.

The proof of this assertion is contained in (1), p. 239.

Proof of Theorem 1. Consider the sum

$$S' = \sum_{x=1}^{p^m} \exp 2\pi i \frac{a(x + bp^m)^n}{p^t}, \quad (1)$$

where $(a, p) = 1$, $r = t/m$, $1 \leq r \leq 1/4n$, and obtain an asymptotic formula for it.

Let $t = t_1n + t_2$, $0 \leq t_2 \leq n - 1$. We split the sum (1) into $t_1 - \delta_n(t) + 1$ sums, collecting together the terms in x not divisible by p , divisible by p but not by p^2 , divisible by p^2 but not by p^3 , and so on. According to this decomposition we obtain the equality

$$S' = p^{m-t_1+\delta_n(t)-1} + \sum_{\nu=0}^{t_1-\delta_n(t)} S'_\nu,$$

where

$$S'_\nu = \sum_{\substack{x=1 \\ (x,p)=1}}^{p^{m-\nu}} \exp 2\pi i \frac{a(x + bp^{m-\nu})^n}{p^{t-\nu n}}, \quad \nu = 0, 1, \dots, t_1 - \delta_n(t).$$

We split the sum over ν into two:

- a) $0 \leq \nu \leq \nu'_0 = \left\lfloor \frac{t}{n-1} \left(1 - \frac{1}{r}\right) \right\rfloor$;
- b) $\nu'_0 + 1 \leq \nu \leq t_1 - \delta_n(t)$.

Consider case a). In the sum S'_ν we make a linear change of the summation variable of the form

$$x = py + z, \quad 1 \leq z \leq p-1, \quad 0 \leq y \leq p^{m-\nu-1} - 1.$$

Then we have the equality

$$x^n = (py + z + bp^{m-\nu})^n = a_0 + a_1py + \dots + a_np^n y^n, \quad (a_\nu, p) = 1, \\ \nu = 0, 1, 2, \dots, n.$$

For the modulus of the sum S'_ν we obtain the estimate:

$$|S'_\nu| \leq p \left| \sum_{y=0}^{p^{m-\nu-1}-1} \exp 2\pi i \frac{a_1 y + a_2 p y^2 + \dots + a_n p^{n-1} y^n}{p^{t-\nu n-1}} \right|.$$

Defining the quantity r_ν from the equality

$$(p^{m-\nu-1})^{r_\nu} = p^{t-\nu n-1},$$

we obtain

$$r_\nu = \frac{t - \nu n - 1}{m - \nu - 1} = r \frac{t - \nu n - 1}{t - \nu r - r}, \quad 0 \leq \nu \leq \nu'_0.$$

It is easy to show that $1 \leq r_\nu \leq 2r$, $\nu = 0, 1, \dots, \nu'_0$. Applying the estimate of the lemma, after simple computations we arrive at the estimate

$$|S'_\nu| = O(P^{1-\gamma/r^2}).$$

Consider case b). We have

$$S'_\nu = \sum_{\substack{x=1 \\ (x,p)=1}}^{p^{m-\nu}} \exp 2\pi i \frac{a(x + bp^{m-\nu})^n}{p^{t-\nu n}}, \quad \nu'_0 + 1 \leq \nu \leq t_1 - \delta_n(t).$$

For the selected values of ν the inequality holds:

$$m - \nu \geq t - \nu n, \quad \nu = \nu'_0 + 1, \dots, t_1 - \delta_n(t).$$

Therefore, using the property of a complete trigonometric sum, we obtain

$$S'_\nu = p^{m-\nu-t+\nu n} \sum_{\substack{x=1 \\ (x,p)=1}}^{p^{t-\nu n}} \exp 2\pi i \frac{ax^n}{p^{t-\nu n}}, \quad \nu'_0 + 1 \leq \nu \leq t_1 - \delta_n(t).$$

If $\delta_n(t) = 0$, then

$$l = t - \nu n \geq t - t_1 n = t_2 \geq 1.$$

If $t_2 \geq 2$, then $l \geq 2$ and

$$\sum_{\substack{x=1 \\ (x,p)=1}}^{p^l} \exp 2\pi i \frac{ax^n}{p^l} = 0.$$

If, however, $t_2 = 1$, then $S'_\nu = 0$ for all $\nu = \nu'_0 + 1, \dots, t_1 - 1$, and

$$S'_{t_1} = p^{m-t_1-1} \sum_{x=1}^{p-1} \exp 2\pi i \frac{ax^n}{p}.$$

If $\delta_n(t) = 1$, then $t_2 = 0$,

$$t = nt_1, \quad \nu'_0 + 1 \leq \nu \leq t_1 - 1; \quad t - \nu n \geq n \geq 2;$$

$$S'_\nu = 0, \quad \nu = \nu'_0 + 1, \dots, t_1 - 1.$$

Consequently,

$$S' = p^{m-t_1-1} \delta_n(t-1) \sum_{x=1}^{p-1} \exp 2\pi i \frac{ax^n}{p} + p^{m-t_1-1+\delta_n(t)} + O(P^{1-\gamma/r^2}).$$

2. The equality is obvious

$$S = \sum_{\nu=0}^{s-1} a_\nu p^{-a_\nu+1} \left(\sum_{x=1}^{p^{s-\nu}-1} \exp 2\pi i \frac{a(x + b_m p^{s-\nu-1})^n}{p^t} \right) + O(p),$$

where b_m are certain integers.

For $\nu = 0, 1, \dots, \nu_0 = [s - 1 - 4t/n]$, for each of the inner sums there holds the asymptotic formula which we derived above. If $\nu > \nu_0$, then, estimating each of

the remaining sums trivially, we obtain a remainder term of order $O(P^{1-\gamma/r^2})$. Thus the theorem is completely proved.

Corollary. For the sum S the inequalities hold

$$S \ll \begin{cases} P^{1-\gamma r/n}, & \text{if } 1 \leq r \leq \sqrt[3]{n}, \\ P^{1-\gamma/r^2}, & \text{if } \sqrt[3]{n} \leq r \leq 0.1n. \end{cases}$$

The proof follows from Theorem 1 and simple calculations. Applying the estimates obtained, we get the assertion (see (2)).

Theorem 2. Consider the congruence

$$x_1^n + \dots + x_k^n \equiv y_1^n + \dots + y_k^n \pmod{p^t},$$

$$1 \leq x_\nu, y_\nu \leq P, \quad \nu = 1, 2, \dots, k.$$

For the number I of solutions of this congruence, the following asymptotic formula holds for $k \geq c_1 n$:

$$I = \psi(p, k) P^{2k} p^{-t} + O(P^{2k-1} p^{-t}),$$

where

$$\psi(p, k) = \sum_{\nu=0}^{\infty} p^{-2k\nu} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^\nu} \left| \sum_{x=1}^{p^\nu} \exp 2\pi i \frac{ax^n}{p^\nu} \right|^{2k},$$

and the constant in O depends only on k and t .

This result cannot be improved, since it is easy to prove the inequality

$$I \gg P^{2k} p^{-t} p^{n-2k}.$$

Theorem 1 can be extended to a broader class of trigonometric sums.

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