

# ON THE THEORY OF FREDHOLM EQUATIONS OF THE FIRST KIND

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## **ON THE THEORY OF FREDHOLM EQUATIONS OF THE FIRST KIND**

*(Presented by Academician M. A. Lavrent'ev on 19 VI 1965)*

A number of problems of mathematical physics reduce to Fredholm integral equations of the first kind. Some of the simplest conditions for the solvability of these equations were first found by É. Picard <sup>(1)</sup>.

The purpose of the present work is to study the equation

$$Ax = f, \quad (1)$$

where  $f$  is a given element of a Hilbert space  $H$ ;  $A$  is a completely continuous linear operator mapping  $H$  into itself.

Below we give necessary and sufficient conditions for the solvability of equation (1) and conditions for the normal solvability of equation (1) (in the sense of Hausdorff <sup>(2)</sup>). In the case of an integral operator  $A$  with a positive symmetric kernel, a necessary and sufficient condition for the normal solvability of equation (1) in the class  $C$  of continuous functions is given in Theorem 8.

**Theorem 1.** Let  $B$  be a bounded linear operator mapping  $H$  into itself;  $B^*$  the operator adjoint to  $B$ .

The equations  $Bx = f$  and  $B^*Bx = B^*f$  are equivalent if

$$(f, \omega) = 0$$

for every solution  $\omega$  of the homogeneous adjoint equation

$$B^*\omega = 0.$$

**Theorem 2.** The orthonormal system of solutions of the equation

$$Az = 0 \quad (2)$$

and of eigenvectors of the equation

$$x - \lambda A^* A x = 0 \quad (3)$$

is complete in the space  $H$ .

**Corollary.** The equation

$$Az = 0$$

has nonzero solutions if and only if the system of eigenvectors  $\{x_i\}$  of equation (3) is incomplete in the space  $H$ .

On the basis of the relations between the eigenvectors and eigenvalues of the equations  $x - \lambda A^* A x = 0$ ,  $y - \lambda A A^* y = 0$  <sup>(3)</sup> and the Hilbert-Schmidt expansion for the operator  $A^* A$  (or the generalized Hilbert-Schmidt expansion for the operator  $A$  <sup>(3)</sup>), one obtains

**Theorem 3.** Equation (1) is solvable if and only if\*

$$(f, \omega) = 0 \quad (4)$$

\* A condition equivalent to (4) was indicated by É. I. Zverovich and G. S. Litvinchuk in <sup>(4)</sup>.

for any solution  $\omega$  of the equation  $A^* \omega = 0$ , and

$$\sum_i \lambda_i |d_i|^2 < \infty, \quad (5)$$

where  $\{\lambda_i\}$  are eigenvalues,  $\{y_i\}$  is an orthonormal system of eigenelements of the equation

$$y - \lambda A A^* y = 0, \quad (6)$$

and  $d_i$  are the Fourier coefficients of the element  $f$  with respect to  $\{y_i\}$ .

**Remark 1.** When the operator  $A$  is self-adjoint, in Theorem 3 equation (6) may be replaced by the equation

$$y - \lambda A y = 0$$

and, correspondingly, condition (5) is replaced by the condition

$$\sum_i \lambda_i^2 |d_i|^2 < \infty.$$

**Remark 2.** If  $A^*\omega = 0$  has only the trivial solution, then Theorem 3 becomes E. Picard's theorem (<sup>1</sup>).

**Remark 3.** The equation

$$Bx = f, \quad (7)$$

where  $B$  is a linear bounded operator, after multiplication by a completely continuous operator  $A_1$  having no zeros, is reduced to the form (1)

$$Ax = A_1 f,$$

where  $A = A_1 B$ . Thus a solvability condition for equation (7) is established. The operator  $A_1$  is easily constructed.

In what follows, a normally solvable operator equation will be called a **Hausdorff equation**. Let us note that if, for all  $f$  satisfying condition (4), condition (5) is also fulfilled, then equation (1) is a Hausdorff equation. If among all elements  $f$  satisfying condition (4) there exists at least one for which condition (5) is violated, then equation (1) is not Hausdorff. Hence it follows:

**Theorem 4.** *In order that equation (1) be a Hausdorff equation, it is necessary and sufficient that the operator  $A$  be finite-dimensional.*

Let the equation be given

$$\int_{\Omega} K(P, Q) \varphi(Q) dQ = f(P), \quad (8)$$

where  $\Omega$  is an  $n$ -dimensional domain in Euclidean space  $E^n$ ,

$$K(P, Q) \in L_2(\overline{\Omega} \times \overline{\Omega}), \quad \varphi(Q) \in L_2(\overline{\Omega}), \quad f(P) \in L_2(\overline{\Omega}).$$

**Theorem 5.** *Equation (8) is Hausdorff if and only if the kernel  $K(P, Q)$  is degenerate.*

A direct consequence of Theorems 2 and 4 is

**Theorem 6.** *The equation  $Ax = f$  is not Noetherian.*

Now consider the equation

$$K\varphi \equiv \int_{\Omega} K(P, Q) \varphi(Q) dQ = f(P), \quad (9)$$

where  $\Omega$  is a finite  $n$ -dimensional domain in  $E^n$ ,  $K(P, Q) \in C(\overline{\Omega} \times \overline{\Omega})$ ,  $\varphi(Q) \in C(\overline{\Omega})$ ,  $f(P) \in C(\overline{\Omega})$ . The homogeneous equation adjoint to (9) will be called

equation

$$K'\psi \equiv \int_{\Omega} K(Q, P)\psi(Q) dQ = 0. \quad (10)$$

We shall call equation (9) a Hausdorff equation if, for its solvability, it is necessary and sufficient that the conditions

$$\int_{\Omega} f(P)\psi(P) dP = 0$$

be satisfied for all solutions of equation (10) continuous in  $\overline{\Omega}$ .

**Theorem 7.** Equation (9) is Hausdorff if the kernel  $K(P, Q)$  is degenerate.

**Theorem 8.** If the kernel  $K(P, Q)$  of equation (9) is symmetric and positive in the sense that  $(K\varphi, \varphi) \geq 0$ , then the degeneracy of  $K(P, Q)$  is a necessary and sufficient condition for this equation to be Hausdorff.

The proof of this theorem is based on Mercer's theorem on the uniform convergence of the Schmidt series

$$K(P, Q) = \sum_i \frac{x_i(P)\overline{x_i(Q)}}{\lambda_i}$$

in the domain  $\overline{\Omega} \times \overline{\Omega}$ .

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## REFERENCES

1. É. Goursat, *Course of Mathematical Analysis*, 3, Moscow—Leningrad, 1934.
2. F. Hausdorff, *Set Theory*, Moscow—Leningrad, 1937, p. 266.
3. Sh. I. Mogilevskii, *Izv. vyssh. uchebn. zaved., ser. matem.*, No. 3, 183 (1958).
4. É. I. Zverovich, G. S. Litvinchuk, *Izv. AN SSSR, ser. matem.*, 28, No. 5 (1964).

*Note: Figure translations are in progress. See original paper for figures.*

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