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Abstract

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MATHEMATICS

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DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DERIVATIVES AND STOCHASTIC EQUATIONS FOR GENERALIZED RANDOM PROCESSES

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1°. Let $\mathfrak{h}$ be a Hilbert space, and let T be a positive definite operator with dense domain of definition D in $\mathfrak{h}$, such that the operator T^{-1} is a Hilbert–Schmidt operator.

We transform D into a Hilbert space $\mathfrak{h}_+$, introducing in it the positive norm $\|\varphi\|_+ = \|T\varphi\|$. Let $\mathfrak{h}_- \supset \mathfrak{h}$ be the corresponding space with negative norm, defined by the formula $\|\xi\|_- = \|T^{-1}\xi\|$ for $\xi \in \mathfrak{h}$, obtained from $\mathfrak{h}$ by completion in this norm. (The operator T^{-1} is extended to all of $\mathfrak{h}_-$ so that the last formula remains valid also for $\xi \in \mathfrak{h}_-$ (for details see (1–3).))

We shall consider nonlinear functionals $F(x)$ in $\mathfrak{h}$, admitting an extension to $\mathfrak{h}_-$ and twice differentiable there in the sense that the representation

$$F(x+h) = F(x) + F_1(x;h) + \frac{1}{2}F_2(x;h,h) + o(\|h\|_-^2)$$

is valid for $x, h \in \mathfrak{h}_-$, where $F_1(x;h)$ is a linear, and $F_2(x;h_1, h_2)$ a bilinear, functional in $\mathfrak{h}_-$ for each $x \in \mathfrak{h}_-$. It is easy to see that $F_1(x;h) = (F'(x), h)$, $F_2(x;h_1, h_2) = (F''(x)h_1, h_2)$, where $h, h_1, h_2 \in \mathfrak{h}_-$; $F'(x) \in \mathfrak{h}_+$; $F''(x)$ is a linear operator acting from $\mathfrak{h}_-$ to $\mathfrak{h}_+$ and having finite trace. We shall call $F'(x)$ and $F''(x)$, respectively, the first and second derivatives of the functional $F(x)$. Functionals for which these derivatives are continuous at each point $x \in \mathfrak{h}_-$ in the corresponding norms will be called functionals of class C_2 . For functionals of this class we introduce differential operators of the first and second orders by the formulas

$$L_1(F) = (F'(x), a(x)), \quad L_2(F) = \text{Sp}[B(x)F''(x)],$$

where $a(x) \in \mathfrak{h}_-$, $B(x)$ is a bounded linear operator in $\mathfrak{h}$.

The operator $L_2(F)$ is naturally regarded as elliptic when the operator $\text{Re } B(x)$ is sign-definite, and as the Laplace operator when $B(x) = I$.

2°. The purpose of this article is to consider equations of diffusion type

$$\partial F / \partial t = \text{Sp}[B(x, t)F''] + (F', a(x, t)) + v(x, t)F. \quad (1)$$

Here the operator $B(x, t)$ is assumed positive definite.

We first consider the simpler equation

$$\partial F / \partial t = \text{Sp}[BF''] \quad (2)$$

with constant operator B .

Let P be the operator of orthogonal projection onto a finite-dimensional subspace $\mathcal{L} \subset \mathfrak{h}_+$. We introduce on the cylindrical sets of the space $\mathfrak{h}_-$ with base in $\mathcal{L}$ the Gaussian measure:

$$\mu_P(x, t; S) = [(2\pi t)^n \det(PBP)]^{-1/2} \int_{PS} \exp \left\{ -\frac{1}{2t} ([PBP]^{-1}(y-x), (y-x)) \right\} dy.$$

By virtue of a known result of R. A. Minlos (see ⁽⁴⁾), in the space $\mathfrak{H}_-$, on the σ -ring $\mathfrak{R}$ containing all cylindrical sets, there exists a Gaussian measure $\mu(x, t; S)$ with mean value x and correlation operator Bt , which is an extension of the measures μ_P .

Theorem 1. Let $\Phi(x)$ be a continuous functional on $\mathfrak{H}_-$ satisfying the condition

$$|\Phi(x)| \leq C_\alpha e^{\alpha(B^{-1}x, x)} \quad (C_\alpha > 0, \alpha > 0). \quad (3)$$

There exists an interval $(0, t)$ on which the Cauchy problem for equation (2) with initial condition $F(x, 0) = \Phi(x)$ has a solution representable in the form

$$F(x, t) = \int_{\mathfrak{H}_-} \Phi(y) \mu(x, t; dy). \quad (4)$$

This solution is unique in the class of functionals from C_2^- which, together with the derivative $\partial F / \partial t$, satisfy condition (3). If condition (3) is fulfilled for every $\alpha > 0$, then these assertions are valid for arbitrarily large T .

3°. Let $C(\mathfrak{H}_-)$ be the space of continuous functions $\xi(t)$ ($0 \leq t \leq \tau$) with values in $\mathfrak{H}_-$. By $C_{x, \tau}(\mathfrak{H}_-)$ we denote its subspace consisting of functions satisfying the condition $\xi(\tau) = x$. Let q ($0 < \tau_1 < \dots < \tau_s < \tau$) be a partition of the interval $(0, \tau)$, and let M_q be a cylindrical set in $C_{x, \tau}(\mathfrak{H}_-)$, defined by the partition q and by a collection of sets $M_j \subset \mathfrak{H}_-$ ($j = 1, \dots, s$):

$$M_q = \{\xi(t) : \xi(\tau_j) \in M_j, j = 1, 2, \dots, s\}.$$

On the ring of such cylindrical sets introduce the measure

$$m(M_q) = \int_{\mathfrak{H}_- M_q} \Phi(y) \mu(y_1, \tau_1, dy) \mu(y_2, \tau_2 - \tau_1, dy_1) \cdots \mu(x, \tau - \tau_s, dy_s),$$

where $\Phi(y)$ is a functional possessing the same properties as in Theorem 1. Using the results from ⁽⁵⁾, one can show that this measure can be extended to some σ -ring in $C_{x,\tau}(\mathfrak{H}_-)$ containing all cylindrical sets.

Theorem 2. Let $V(x, t)$ be a bounded, continuous functional on $\mathfrak{H}_-$. The solution of the Cauchy problem

$$\partial F / \partial t = \text{Sp}[BF''] + V(x, t)F, \quad F(x, 0) = \Phi(x),$$

is representable in the form of the integral

$$F(x, t) = \int_{C_{x,\tau}(\mathfrak{H}_-)} \exp \left\{ \int_0^\tau v(\xi(u), u) du \right\} dm.$$

4°. To construct a solution of equation (1) one can apply the theory of stochastic equations developed by I. I. Gikhman, K. Ito, and A. V. Skorokhod (see ⁽⁶⁾). V. V. Baklan ⁽⁷⁾ and T. L. Chantladze ⁽⁸⁾ considered stochastic equations in Hilbert space and, by this method, constructed the solution of equations in functional derivatives of a certain special form, corresponding to the case when the operator B in (1) has finite trace. Below we make essential use of the results from ^(7, 8) in order to investigate a more general stochastic equation in a space with negative norm. The solution of such an equation is, generally speaking, a generalized random process.

Thus, we shall consider random variables with values in the space $\mathfrak{H}_-$. In the case when $\mathfrak{H}$ is a space of functions, they are generalized random functions (processes or fields).

Let $w(t)$ be a homogeneous random process with independent increments, taking values in $\mathfrak{H}_-$ and possessing the following property: for any collection $\varphi_j \in \mathfrak{H}_+$ ($j = 1, \dots, n$), the n -dimensional random

the quantity $(w(t_1) - w(t_2), \varphi_j)$ is a Gaussian random variable with zero mean and correlation matrix $(t_1 - t_2) \|(\varphi_k, \varphi_j)\|_{j,k=1}$.

Under the usual assumptions, stochastic integrals of the form

$$\int_0^\tau A(t) dw(t), \quad \int_0^\tau (\alpha(t), dw(t)),$$

are introduced in the usual way, where $A(t), \alpha(t)$ are random functions whose values are, respectively, operators in $\mathfrak{H}_-$ and vectors in $\mathfrak{H}_-$. These integrals obey the estimates

$$\begin{aligned} \mathbf{M} \left\| \int_0^\tau A(t) dw(t) \right\|_-^2 &\leq (\text{Sp } T^{-2}) \cdot \mathbf{M} \left\| \int_0^\tau (T^{-1}AT)^*(T^{-1}AT) dt \right\| \leq \\ &\leq (\text{Sp } T^{-2}) \int_0^\tau \mathbf{M} \|A(t)\|^2 dt, \\ \mathbf{M} \left| \int_0^\tau (\alpha(t), dw(t)) \right|^2 &\leq \int_0^\tau \mathbf{M} \|\alpha(t)\|^2 dt. \end{aligned}$$

Consider the stochastic equation

$$\xi(u) = \xi(t) + \int_t^u a(\xi(s), s) ds + \int_t^u A(\xi(s), s) dw(s) \quad (0 \leq t \leq u \leq \tau), \quad (5)$$

where $\xi(s) \in \mathfrak{H}_-$, and $a(\xi, s)$ and $A(\xi, s)$ are measurable functions on $\mathfrak{H}_- \times [0, \tau]$, whose values are, respectively, vectors from $\mathfrak{H}_-$ and bounded operators acting in $\mathfrak{H}_-$.

Theorem 3. *Suppose that the functions $a(\xi, s)$ and $A(\xi, s)$ satisfy in $\mathfrak{H}_-$ a Lipschitz condition with a constant independent of ξ and s .*

If $\mathbf{M} \|\xi(t)\|_-^2 < \infty$, then there exists a unique, up to stochastic equivalence, solution of equation (5) satisfying the condition

$$\int_t^\tau \|\xi(u)\|_-^2 du < \infty.$$

This solution is a Markov process with values in $\mathfrak{H}_-$.

The proof of the theorem is based on the expansion, convergent in $\mathfrak{H}_-$,

$$w(t) = \sum_{k=1}^{\infty} w_k(t) \varphi_k$$

in the eigenvectors φ_k of the operator T . Here

$$w_k(t) = (\varphi_k, w(t)) \quad (k = 1, 2, \dots)$$

is a sequence of mutually independent Wiener processes.

Under some additional assumptions on the smoothness of the coefficients a and A , one can show that the backward Kolmogorov equation for the process $\xi(t)$ satisfying equation (5) is an equation of type (1),

$$\partial F / \partial t + \text{Sp}(A^* F'' A) + (F', a) = 0. \quad (6)$$

In this case the solution of the Cauchy problem for equation (5) is expressed in terms of integrals with respect to the measure determined by the random process $\xi(t)$.

Theorem 4. Suppose that the coefficients $a(\xi, t)$, $A(\xi, t)$, and the functional $\Phi(\xi)$ have, in $\mathfrak{H}_-$, derivatives with respect to ξ up to and including second order, satisfying the Lipschitz condition, and let $\mu(x, t, u, S)$ be the measure generated in $\mathfrak{H}_-$ by the solution of equation (5) with $\xi(t) = x$.

Then the functional

$$F(x, t) = \int_{-\mathfrak{H}}^{\mathfrak{H}} \Phi(y) \mu(x, t, u, dy)$$

satisfies, for $t \leq u$, equation (6) and the condition $F(x, u) = \Phi(x)$.

Changing the sign before t , one easily obtains the formula also for the solution of an equation of type (1) in the domain $t \geq u$.

5°. Consider a pair of random processes ξ_1 and ξ_2 satisfying equations of type (5) with one and the same operator A . Under certain conditions the probability measures corresponding to these processes are mutually absolutely continuous. The formula for the density obtained in ^(6, 9) is also generalized (see also ⁽³⁾, where the measures were constructed not with the aid of stochastic equations, but with the aid of differential equations for probability distributions). For simplicity of formulation, suppose that $\xi_1(t) = \xi_2(t)$. Then the following result is valid.

Theorem 5. Suppose that the condition is satisfied

$$\alpha(\xi, u) = A^{-1}(\xi, u) [a_2(\xi, u) - a_1(\xi, u)] \in \mathfrak{H} \quad (\xi \in \mathfrak{H}, -t \leq u \leq \tau).$$

Then the measures m_1, m_2 , corresponding to the random processes ξ_1, ξ_2 , are mutually absolutely continuous and the formula is valid

$$\log \frac{dm_2}{dm_1}[\xi_1(u)] = -\frac{1}{2} \int_t^\tau \|\alpha(\xi_1(u), u)\|^2 du + \int_t^\tau (\alpha(\xi_1(u), u), dw(u)).$$

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