

ON A VARIATIONAL PROBLEM IN HILBERT SPACE AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

Z. I. Khalilov, È. D. Aslanov

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.80606>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.221+517.216.2

MATHEMATICS

Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. KHALILOV, E. J. ASLANOV

ON A VARIATIONAL PROBLEM IN HILBERT SPACE AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

1. Let H_1 and H_2 be real Hilbert spaces with scalar products $(x, y)_1$ and $(\xi, \eta)_2$; let K be a linear continuous operator acting from H_1 into H_2 ; and let $E(x)$ be a quadratic functional in H_1 , defined by the formula

$$E(x) = (Kx - \xi, Kx - \xi)_2 + (x, x)_1. \quad (1)$$

Then the following holds.

Theorem. For every $\xi \in H_2$ there exists $x_0 \in H_1$ minimizing the functional (1), which is determined uniquely from the equation

$$x + K^*Kx = K^*\xi. \quad (2)$$

Proof. Obviously, equation (2) has, moreover, a unique solution $x_0 \in H_1$. Therefore, for any $y \in H_1$ we have

$$\begin{aligned} E(x_0 + y) &= E(x_0) + 2(K^*Kx_0 + x_0 - K^*\xi, y)_1 + (Ky, Ky)_2 + (y, y)_1 \\ &= E(x_0) + (Ky, Ky)_2 + (y, y)_1. \end{aligned} \quad (3)$$

It follows from (3) that $E(x_0) < E(x_0 + y)$ for every $y \neq 0$, i.e. $x_0 \in H_1$ is the unique element minimizing the functional.

2. Let H be a real Hilbert space with scalar product (φ, ψ) . Then the aggregate of vector functions with values in H , defined on the interval $[0, T]$, with scalar product

$$[f, g] = \int_0^T (f(s), g(s)) ds,$$

will be a Hilbert space (1), which we shall denote by $L_2 = L_2(H, [0, T])$.

Take as H_1 the set of pairs $\langle f, \varphi \rangle$, $f \in L_2$, $\varphi \in H$, with scalar product of the elements $x = \langle f, \varphi \rangle$ and $y = \langle g, \psi \rangle$, defined by the formula

$$(x, y)_1 = a[f, g] + b(\varphi, \psi), \quad a > 0, \quad b > 0.$$

We define the operator K by the formula

$$Kx = Bf + U\varphi,$$

where B and U are linear continuous operators acting from L_2 into H_2 and from H into H_2 , respectively. The operator K^* , acting from H_2 into H_1 , will have the form

$$K^*\xi = \langle a^{-1}B^*\xi, b^{-1}U^*\xi \rangle.$$

Hence

$$K^*Kx = \langle a^{-1}B^*Bf + a^{-1}B^*U\varphi, b^{-1}U^*Bf + b^{-1}U^*U\varphi \rangle,$$

and equation (2) is written in the form of the system

$$\begin{aligned} B^*Bf + af + B^*U\varphi &= B^*\xi, \\ U^*U\varphi + b\varphi + U^*Bf &= U^*\xi, \end{aligned} \tag{4}$$

whose solution minimizes the functional

$$E(f, \varphi) = (Bf + U\varphi - \xi, Bf + U\varphi - \xi)_2 + a \int_0^T \|f(s)\|^2 ds + b\|\varphi\|^2. \tag{5}$$

3. The constructions given above can be applied to the solution of the following variational problems. Let

$$u(t) = U(t, 0)\varphi + \int_0^t U(t, s)f(s) ds$$

be a generalized solution of the Cauchy problem

$$du(t)/dt + A(t)u(t) = f(t), \quad 0 < t \leq T, \quad u(0) = \varphi, \tag{6}$$

in the space H , where $A(t)$ is, generally speaking, an unbounded operator in H . Here $U(t, s)$ is an operator-valued function, continuous in the aggregate of

variables for $0 \leq s \leq t \leq T$, with values in the ring of linear continuous operators in H (for conditions under which $A(t)$ can generate $U(t, s)$, see ⁽²⁻⁴⁾).

Problem 1. Find a pair $f \in L_2$, $\varphi \in H$, minimizing the functional

$$E_1(f, \varphi) = \|u(T) - \psi\|^2 + a \int_0^T \|f(s)\|^2 ds + b\|\varphi\|^2, \quad a > 0, \quad b > 0, \quad (7)$$

where $\psi \in H$ is a given element.

The functional E_1 may be regarded as a special case of the functional (5), if one sets $H_2 = H$,

$$Bf = \int_0^T U(T, s)f(s) ds, \quad U\varphi = U(T, 0)\varphi.$$

Then $U^*\varphi = U^*(T, 0)\varphi$, $B^*\varphi = U^*(T, t)\varphi$.

System (4) in this case takes the form

$$\begin{aligned} af(t) + \int_0^T U^*(T, t)U(T, s)f(s) ds + U^*(T, t)U(T, 0)\varphi &= U^*(T, t)\psi, \\ b\varphi + U^*(T, 0)U(T, 0)\varphi + \int_0^T U^*(T, 0)U(T, s)f(s) ds &= U^*(T, 0)\psi. \end{aligned} \quad (8)$$

Putting $t = 0$ in the first equation and comparing with the second, we obtain $b\varphi = af(0)$, where $f(t)$ is determined from the loaded integral equation

$$af(t) + \int_0^T U^*(T, t)U(T, s)f(s) ds + \frac{a}{b}U^*(T, t)U(T, 0)f(0) = U^*(T, t)\psi, \quad (9)$$

Problem 2. Find a pair $f \in L_2$, $\varphi \in H$, minimizing the functional

$$E_2(f, \varphi) = \int_0^T \|U(s) - g(s)\|^2 ds + a \int_0^T \|f(s)\|^2 ds + b\|\varphi\|^2, \quad a > 0, \quad b > 0, \quad (10)$$

where $g \in L_2$ is a given function.

The functional E_2 may be regarded as a special case of the functional (5), if one sets $H_2 = L_2$, $U\varphi = U(t, 0)\varphi$,

$$Bf = \int_0^t U(t, s)f(s) ds.$$

Then

$$U^*f = \int_0^T U^*(s, 0)f(s) ds, \quad B^*f = \int_t^T U^*(s, t)f(s) ds.$$

System (4) in this case takes the form:

$$\begin{aligned} \int_t^T \int_0^s U^*(s, t)U(s, \tau)f(\tau) d\tau ds + af(t) + \int_t^T U^*(s, t)U(s, 0)\varphi ds = \\ = \int_t^T U^*(s, t)g(s) ds, \\ \int_0^T \int_0^s U^*(s, 0)U(s, \tau)f(\tau) d\tau ds + b\varphi + \int_0^T U^*(s, 0)U(s, 0)\varphi ds = \\ = \int_0^T U^*(s, 0)g(s) ds. \end{aligned}$$

Just as in problem 1, we obtain $b\varphi = af(0)$, where $f(t)$ is determined from the loaded integral equation

$$\begin{aligned} af(t) + \int_t^T \int_0^s U^*(s, t)U(s, \tau)f(\tau) d\tau ds + \\ + \frac{a}{b} \int_t^T U^*(s, 0)U(s, 0)f(0) ds = \int_t^T U^*(s, t)g(s) ds. \end{aligned} \quad (11)$$

The problems set forth, following Bellman ⁽⁵⁾, may be regarded as problems of optimal control. The first term in the functionals E_1 and E_2 represents the cost of deviation, and the other two terms the cost of control, with a and b determining the weight assigned to the controls f and φ .

4. Many nonstationary problems for partial differential equations can be written in the form (6). Therefore the solution of variational problems 1 and 2 can be applied to the investigation of analogous questions for partial differential equations. As an application, consider the Cauchy problem for the heat equation

$$\partial u(t, x)/\partial t - \partial^2 u(t, x)/\partial x^2 = f(t, x), \quad -\infty < x < \infty, \quad t \geq 0,$$

$$u(0, x) = \varphi(x).$$

Here $H = L_2(-\infty, \infty)$, $A(t)$ does not depend on t and is determined by the equality $Au = -\partial^2 u/\partial x^2$, $D(A) = W_2^2(-\infty, \infty)$. Then A is a self-adjoint positive operator in $L_2(-\infty, \infty)$, and $U^*(t, s) = U(t, s) = \exp(-(t-s)A)$, where (see (6))

$$\exp(-tA)\varphi = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\eta)^2}{4t}\right) \varphi(\eta) d\eta, \quad t > 0.$$

The functionals E_1 and E_2 take, for this case, the form

$$E_1(f, \varphi) = \int_{-\infty}^{\infty} |u(T, x) - \psi(x)|^2 dx + a \int_0^T \int_{-\infty}^{\infty} |f(t, x)|^2 dx dt + b \int_{-\infty}^{\infty} |\varphi(x)|^2 dx,$$

$$E_2(f, \varphi) = \int_0^T \int_{-\infty}^{\infty} |u(t, x) - g(t, x)|^2 dx dt + a \int_0^T \int_{-\infty}^{\infty} |f(t, x)|^2 dx dt + b \int_{-\infty}^{\infty} |\varphi(x)|^2 dx.$$

Problem 1 becomes the problem of the best approximation to a prescribed temperature $\psi(x)$ over a definite time interval T , and its solution is found from the nonhomogeneous integral equation

$$\begin{aligned} af(t, x) + \int_0^T \int_{-\infty}^{\infty} K(x - \eta, t, s) f(s, \eta) d\eta ds + \\ + \frac{a}{b} \int_{-\infty}^{\infty} K(x - \eta, t, 0) f(0, \eta) d\eta = G(t, x), \end{aligned} \quad (12)$$

$$K(\xi, t, s) = \frac{1}{2\sqrt{\pi(2T-s-t)}} \exp\left(-\frac{\xi^2}{4(2T-s-t)}\right);$$

$$G(t, x) = \int_{-\infty}^{\infty} \frac{\exp(-(x-\eta)^2/4(T-t))}{2\sqrt{\pi(T-t)}} \psi(\eta) d\eta.$$

Problem 2 becomes the problem of the best approximation to a prescribed temperature regime $g(t, x)$ over a definite time interval T , and its solution is found from equation (12), where

$$K(\xi, t, s) = \sqrt{\frac{2T-s-t}{\pi}} \exp\left(-\frac{\xi^2}{4(2T-s-t)}\right) - \sqrt{\frac{|t-s|}{\pi}} \times \\ \times \exp\left(-\frac{\xi^2}{4|t-s|}\right) - \frac{\xi}{4} \left[\Phi\left(\frac{\xi}{2\sqrt{2T-s-t}}\right) - \Phi\left(\frac{\xi}{2\sqrt{|t-s|}}\right) \right]; \\ G(x, t) = \int_t^T \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-\eta)^2}{4(s-t)}\right)}{2\sqrt{\pi(s-t)}} g(s, \eta) d\eta ds.$$

Here

$$\Phi(a) = 2\pi^{-1/2} \int_0^a e^{-\tau^2} d\tau$$

is the probability integral.

The present article is, in content, close to the works ^(7,8). Part of the results presented, concerning the problem of the best approximation to a prescribed temperature regime, is contained in ^(9,10).

Institute of Cybernetics
Academy of Sciences of the Azerbaijan SSR

Received
5 IV 1966

REFERENCES

1. E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, IL, 1962.
2. T. Kato, Collection of Translations, *Mathematics*, **2** (4) (1958).
3. P. E. Sobolevskii, Tr. Moscow Math. Soc., **10** (1961).
4. H. Tanabe, *Osaka Math. J.*, **11**, No. 2 (1959).
5. R. Bellman, I. Glicksberg, O. Gross, *Some Questions in the Mathematical Theory of Control Processes*, IL, 1962.
6. E. Hille, *Functional Analysis and Semi-Groups*, IL, 1951.
7. E. I. Halilov, DAN, **155**, No. 4 (1964).
8. E. I. Halilov, Dokl. Acad. Sci. Azerbaijan SSR, **20**, No. 5 (1964).

9. E. J. Aslanov, Dokl. Acad. Sci. Azerbaijan SSR, **21**, No. 7 (1965).
10. E. J. Aslanov, Izv. Acad. Sci. Azerbaijan SSR, **5**, mathematical series (1965).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.