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Abstract

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THEORY OF ELASTICITY

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SOLUTION OF THE SPATIAL AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR MULTIPLY CONNECTED BODIES OF REVOLUTION BY MEANS OF GENERALIZED ANALYTIC FUNCTIONS

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In papers ^(1,2) a method was given for solving spatial axisymmetric problems of the theory of elasticity by means of analytic functions of a complex variable. In paper ⁽³⁾, for this purpose, p -analytic functions were used. The author ⁽⁴⁾, for the case of simply connected bodies, applied generalized analytic functions of a somewhat different class. Below the application of this method to multiply connected bodies is considered.

1. Let D be the plane domain occupied by the meridional section of a body of revolution. The outer boundary L_{n+1} of this domain may consist either of one closed contour intersecting the axis of symmetry oz , or of two closed contours L'_{n+1} (for $r > 0$) and L''_{n+1} (for $r < 0$), situated on both sides of oz . Let L'_k and L''_k ($k = 1, 2, \dots, m$) be symmetrically situated inner contours not intersecting the axis oz , and let L_k ($k = m+1, \dots, n$) be the remaining inner contours. By $L = \sum L_k$ we shall understand the complete boundary of the domain. We assume that all contours have curvature satisfying the condition H (Hölder).

It can be shown ⁽⁴⁾ that the differential equations of equilibrium will be satisfied if the components of the elastic displacement w, u are represented in the form

$$2G(w + iu) = \chi' \Phi(t, \bar{t}) - t\Phi'(t, \bar{t}) - \overline{\Psi(t, \bar{t})} \quad (1)$$

$$(\chi' = 3.5 - 4\nu, \quad t = z + ir).$$

Here ν is Poisson's ratio; G is the shear modulus; $\Phi(t, \bar{t})$ and $\Psi(t, \bar{t})$ are generalized analytic functions satisfying the complex equation of the form $2\partial\Phi/\partial\bar{t} - (\Phi - \bar{\Phi})/(t - \bar{t}) = 0$ and the condition $\Phi(\bar{t}, t) = \overline{\Phi(t, \bar{t})}$. By $\Phi'(t, \bar{t})$ is meant

the derivative in the sense of Bers with respect to the pair $(1, i/r)$ ⁽⁵⁾. For the stresses the formulas will be

$$\sigma_z + \sigma_r + \sigma_\theta = 2(1 + \nu)(\Phi' + \bar{\Phi}'), \quad \sigma_z + \sigma_r = 2(\Phi' + \bar{\Phi}') - \frac{2Gu}{r}, \quad (2)$$

$$\sigma_z + i\tau_{zr} = 1.5\Phi' + \bar{\Phi}' - t\bar{\Phi}'' - \bar{\Psi}'.$$

From the conditions of single-valuedness of the stresses and displacements there follow the representations

$$\Phi(t, \bar{t}) = \Phi_*(t, \bar{t}) + \sum_{k=1}^n A_k \Theta(t, \bar{t}; t_k, \bar{t}_k) + \sum_{k=1}^m B_k \Xi(t, \bar{t}; t_k, \bar{t}_k), \quad (3)$$

$$\Psi(t, \bar{t}) = \Psi_*(t, \bar{t}) - \chi' \sum_{k=1}^n A_k \Theta(t, \bar{t}; t_k, \bar{t}_k) + \chi' \sum_{k=1}^m B_k \Xi(t, \bar{t}; t_k, \bar{t}_k),$$

where A_k and B_k are real constants; t_k are certain arbitrary points situated inside the contours L'_k ($k = 1, 2, \dots, m$) and L_k ($k = m + 1, \dots, n$); $\Phi_*(t, \bar{t})$ and $\Psi_*(t, \bar{t})$ are generalized analytic functions regular in D . The functions

$$\begin{aligned} \Theta(t, \bar{t}; t_k, \bar{t}_k) &= -\frac{1}{|t - \bar{t}|} \int_{\bar{t}}^t \sqrt{\frac{\xi - \bar{t}}{\xi - t}} \frac{d\xi}{\sqrt{(\xi - t_k)(\xi - \bar{t}_k)}}, \\ \Xi(t, \bar{t}; t_k, \bar{t}_k) &= -\frac{1}{|t - \bar{t}|} \int_{\bar{t}}^t \sqrt{\frac{\xi - \bar{t}}{\xi - t}} \frac{2\xi - t_k - \bar{t}_k}{\sqrt{(\xi - t_k)(\xi - \bar{t}_k)}} d\xi - \pi \end{aligned} \quad (4)$$

are analogues of the logarithm: upon traversing the contour L'_k counterclockwise, $\Xi(t, \bar{t}; t_k, \bar{t}_k)$ acquires the increment 2π , while $\Theta(t, \bar{t}; t_k, \bar{t}_k)$ acquires the increment $2\pi/(t - \bar{t})$. These functions can be expressed in terms of elliptic integrals of the first and second kind.

When t coincides with a contour point τ , equality (1) turns into a boundary condition for Φ and Ψ corresponding to the second fundamental problem. For the first fundamental problem one can obtain

$$\begin{aligned} &0.5 \Phi(\tau, \bar{\tau}) + \tau \Phi'(\tau, \bar{\tau}) + \overline{\Psi(\tau, \bar{\tau})} - \\ &- 2(1 - \nu) \int_{\tau_k}^{\tau} \left[\Phi(t, \bar{t}) - \overline{\Phi(t, \bar{t})} - \frac{C'_k}{t - \bar{t}} \right] \frac{d(t + \bar{t})}{t - \bar{t}} - \end{aligned}$$

$$-\frac{2(1-\nu)}{\tau-\bar{\tau}} C'_k - C_k = -R_k + \frac{i}{y} Z_k \quad \text{on } L'_k \quad (k=1, 2, \dots, n+1), \quad (5)$$

where

$$Z_k = \int_{\tau_k}^{\tau} r p_z ds, \quad R_k = \int_{\tau_k}^{\tau} \left(p_r + \frac{1}{r^2} Z_k \frac{dz}{ds} \right) ds, \quad y = \text{Im } \tau.$$

Here τ_k denotes fixed points of the contours L'_k (for $k \geq m+1$ they lie on the axis of symmetry); p_z and p_r are the intensities of the external forces acting on the surface formed by rotating the contour L'_k about the oz -axis; C_k and C'_k are real constants, and it is known for them that

$$C'_k = 4\pi \sum_{j=m+1}^k A_j \quad (k=m+1, \dots, n), \quad C_{n+1} = C'_{n+1} = 0 \quad (6)$$

(the contours L_k , $k=m+1, \dots, n$, are numbered in the order in which they intersect the axis of symmetry). The coefficients A_k and B_k must satisfy the system of algebraic equations

$$A_k = \frac{1}{4\pi(1-\nu)} \int_{L_k} r p_z ds \quad (k=1, 2, \dots, n), \quad (7)$$

$$4\pi B_k + \int_{L'_k} \left(\Phi - \bar{\Phi} - \frac{1}{t-\bar{t}} C'_k \right) \frac{d(t+\bar{t})}{t-\bar{t}} = \frac{1}{2(1-\nu)} \int_{L'_k} \left(p_r + \frac{1}{r^2} Z_k \frac{dr}{ds} \right) ds$$

$$(k=1, 2, \dots, m). \quad (8)$$

It can be shown that the regular solution of both fundamental problems is unique.

2. Let us obtain integral equations for the solution of the boundary-value problems. In the case of the second fundamental problem we represent the regular part of the expression

of equations (3) in the form of generalized Cauchy-type integrals

$$\Phi_*(t, \bar{t}) = \frac{1}{2\pi i} \int_L F(\tau, \bar{\tau}) W(t, \bar{t}; \tau, \bar{\tau}) d\tau \quad \left(F(\bar{\tau}, \tau) = \overline{F(\tau, \bar{\tau})} \right),$$

$$\Psi_*(t, \bar{t}) = \frac{1}{2\pi i} \int_L \bar{F} \bar{W} [g d\tau + 0.5 d\bar{\tau}] - \frac{1}{2\pi i} \int_L F \left[\bar{\tau} \frac{\partial W}{\partial z} d\tau - W d\bar{\tau} \right], \quad (9)$$

where W is the generalized Cauchy kernel ⁽⁴⁾, $g = -\chi' + 0.5$ for the second fundamental problem and $g = 1$ for the first fundamental problem.

Introduce the notation (τ_0 is a point of the contour)

$$\begin{aligned} \Omega(\tau_0) = & \frac{g}{2\pi i} \int_L F[W d\tau - \bar{W} d\bar{\tau}] - \frac{1}{2\pi i} \int_L \bar{F}\bar{W} \left[d\tau - \frac{\tau - \tau_0}{\bar{\tau} - \bar{\tau}_0} d\bar{\tau} \right] - \\ & - \frac{1}{4\pi i} \int_L F \left[1 + \frac{\bar{\tau} - \tau_0}{\tau - \bar{\tau}_0} \right] [W + \bar{W}] d\tau, \quad W = W(\tau_0, \bar{\tau}_0; \tau, \bar{\tau}). \end{aligned} \quad (10)$$

Substitute (3) into (1). Letting t tend to τ_0 and noting that

$$\Omega(\tau_0) = -gF(\tau_0, \bar{\tau}_0) + (g - 0.5)\Phi_*^+(\tau_0, \bar{\tau}_0) + \tau_0\bar{\Phi}'_*(\tau_0, \bar{\tau}_0) + \Psi_*^+(\tau_0, \bar{\tau}),$$

where on the right-hand side stand the boundary values of the generalized Cauchy-type integrals (9), we obtain the integral equation for the density $F(\tau, \bar{\tau})$

$$\begin{aligned} (\chi' - 0.5)F(\tau_0, \bar{\tau}_0) - \Omega(\tau_0) - \sum_{j=1}^n A_j S_j(\tau_0) - \sum_{j=1}^m B_j T_j(\tau_0) = \\ = 2G(w + in) \quad \text{on } L, \end{aligned} \quad (11)$$

where we put

$$A_j = \operatorname{Re} \int_{L'_j} F(\tau, \bar{\tau}) ds, \quad B_j = \operatorname{Im} \int_{L'_j} F(\tau, \bar{\tau}) ds. \quad (12)$$

By S_j and T_j are denoted known functions containing Θ , Ξ , and their derivatives.

In the case of the first fundamental problem, both expressions (9) should be supplemented by the term

$$\sum_{j=1}^n b_j \Theta'(t, \bar{t}; t_j, \bar{t}_j) \quad \text{with} \quad b_j = \operatorname{Re} \int_{L'_j} \overline{F(\tau, \bar{\tau})} (\tau - \bar{\tau}) d\tau.$$

Substituting (3) into (5) and transferring the known terms to the right, we shall have the integral equation

$$F(\tau_0, \bar{\tau}_0) + \Omega(\tau_0) + \sum_{j=1}^n b_j R_{kj}(\tau_0) + \sum_{j=1}^m B_j T_{kj}(\tau_0) - C'_k P_k(\tau_0) - C_k -$$

$$-(1-\nu) \int_{\tau_k}^{\tau_0} (F - \bar{F}) \frac{d(\tau + \bar{\tau})}{\tau - \bar{\tau}} - \frac{1-\nu}{\pi i} \int_L F Q_k d\tau = f_k(\tau_0) \quad \text{on } L'_k$$

$$(k = 1, 2, \dots, n+1). \quad (13)$$

Here R_{kj} , T_{kj} , and P_k are known functions,

$$Q_k(\tau_0, \bar{\tau}_0; \tau, \bar{\tau}) = \int_{\tau_k}^{\tau_0} [W(t, \bar{t}; \tau, \bar{\tau}) - W(t, \bar{t}; \tau, \bar{\tau})] \frac{d(t + \bar{t})}{t - \bar{t}}. \quad (14)$$

The prescribed functions $f_k(\tau_0)$ are continuous on the contours L'_k , cut at the points τ_k . For the constants C_k and C'_k we take

$$C_k = -\operatorname{Re} \int_{L'_k} F(\tau, \bar{\tau}) ds \quad (k = 1, 2, \dots, n), \quad C_{n+1} = 0, \quad (15)$$

$$C'_k = -\operatorname{Im} \int_{L'_k} F(\tau, \bar{\tau}) ds \quad (k = 1, 2, \dots, m), \quad C'_k = 0 \quad (k \geq m+1).$$

The coefficients B_j can be eliminated by means of (8), which, however, is not necessary in a practical solution.

The integral equations obtained are analogues of D. I. Sherman's equations⁶ for the plane problem. It is easy to show that they reduce to a system of two real Fredholm equations of the second kind. Repeating the arguments of⁶, one can verify the solvability of these equations if their right-hand side is differentiable in the class H .

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Note: Figure translations are in progress. See original paper for figures.

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