

HARMONIC ANALYSIS ON THE ONE-SHEETED HYPERBOLOID

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Abstract

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MATHEMATICS

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HARMONIC ANALYSIS ON THE ONE-SHEETED HYPERBOLOID

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1. Let a bilinear form be given in the n -dimensional real space R^n : $[x, y] = -x_1y_1 + x_2y_2 + \dots + x_ny_n$; let G be the connected group of linear transformations with determinant 1 that preserve this form. G acts transitively on the one-sheeted hyperboloid $X: [x, x] = 1$. The shifts $f(x) \rightarrow f(xg)$ form a unitary representation of G in $L^2(X)$ with respect to the invariant measure $dx = |x_1|^{-1}dx_2 \dots dx_n$. X is an example of a homogeneous space with a noncompact stationary subgroup. Harmonic analysis on such spaces has been studied only in special cases. An analogue of the Plancherel formula on the one-sheeted hyperboloid for $n = 4$ (G is the Lorentz group) was obtained by the horisphere method in ⁽¹⁾ (see also ⁽²⁾).

In the present work $L^2(X)$ is decomposed into irreducibles with the aid of spherical functions.

Let W be the stationary subgroup of the point $x^0 = (0, 1, 0, \dots, 0)$; let T_g be an irreducible unitary representation of G in a Hilbert space H , and let θ be a linear functional defined on an everywhere dense set D and invariant with respect to W . Then the expression $(T_g\theta, \theta)$, understood in the regularized sense (see § 5), defines a function on X : $\Phi(x) = (T_g\theta, \theta)$, $x = x^0g$, which is called spherical. The main problem consists in expanding the δ -function on X in spherical functions.

2. **Representations of G associated with the cone.** Representations of the group G were studied in ⁽³⁻⁵⁾ (see also ⁽⁶⁾, Ch. X, § 2).

Let τ be a complex number; D_τ the space of infinitely differentiable functions $\varphi(\xi)$ on the upper part of the cone $X_0^+ : [x, x] = 0, x_1 > 0$, homogeneous of degree $\sigma = (2-n+\tau)/2$: $\varphi(t\xi) = t^\sigma \varphi(\xi)$, $t > 0$. In D_τ there acts a representation T^τ of the group G : $\varphi(\xi) \rightarrow \varphi(\xi g)$. A function $\varphi \in D_\tau$ is determined by its values on the set $Y : \xi_1 - \xi_2 = 1$; a point $\eta \in Y$ has the form

$$\eta = (\eta_1, \eta_2, \eta) = \left(\frac{1}{2}(\|\eta\|^2 + 1), \frac{1}{2}(\|\eta\|^2 - 1), \eta_3, \dots, \eta_n \right),$$

where $\|\eta\|^2 = \eta_3^2 + \dots + \eta_n^2$. D_τ is realized as a certain aggregate of functions from $C^\infty(Y)$.

The operator

$$A_\tau : \varphi(\eta) \rightarrow \frac{1}{\Gamma(-\tau/2)} \int \|\eta - \eta'\|^{2-n-\tau} \varphi(\eta') d\eta' \quad (d\eta = d\eta_3 \cdots d\eta_n)$$

maps D_τ into $D_{-\tau}$ and commutes with the action of G .

Continuous series. $\tau = i\rho$, $-\infty < \rho < \infty$. T^τ is irreducible and preserves the scalar product

$$(\varphi, \varphi) = \int |\varphi(\eta)|^2 d\eta,$$

T^τ and $T^{-\tau}$ are equivalent.

Discrete series. For $\tau = 2 - n - 2l$, $l = 0, 1, 2, \dots$, the operator A_τ has in D_τ its own null subspace F_τ , which is irreducible with respect to the restriction of T^τ (except for $n = 3$) and in which there exists an invariant scalar product

$$B(\varphi, \varphi) = c_{nl} \int \|\eta - \eta'\|^{2l} \nu_n \|\eta - \eta'\| \varphi(\eta) \overline{\varphi(\eta')} d\eta d\eta'.$$

The constant c_{nl} here is chosen equal to $(-1)^{l+1} 2\pi^{2-n} \Gamma(l+n-2)/\Gamma(l+1)\Gamma(-\tau/2)$. The representations in F'_τ form a discrete series.

D_{ip} and F_{2-n-2l} are completed to Hilbert spaces H_p and H_l with respect to the scalar products already present.

3. Functions invariant with respect to the stationary subgroup. The representation T^τ is extended from D_τ to the space D'_τ of generalized functions $\theta(\eta)$ on X_0^+ , homogeneous of degree σ , $\theta : \varphi(\eta) \rightarrow \int \theta(\eta) A_\tau \varphi(\eta) d\eta$. For $\tau = 2 - n - 2l$ denote by F'_τ the null subspace of the operator A_τ . In D'_τ there exist two linearly independent functions invariant with respect to W , namely $(\xi_2)_\pm^\sigma$, if $\sigma \neq -m$, m natural, and ξ_2^{-m} , $\delta^{(m-1)}(\xi_2)$, if $\sigma = -m$. Only one of the functions ξ_2^{2-n-l} , $\delta^{(n-3+l)}(\xi_2)$ belongs to F'_τ : the first for odd n , the second for even n . Put $\theta_l(\eta) = \beta_l(\eta_2)$, where $\beta(t) = (-1)^{(n-3)/2} t^{2-n-l}$ for odd n , $\beta(t) = (-1)^t l! \Gamma^{-1}(l+n-2) \delta^{(n-3+l)}(t)$ for even n .

4. Integral representation of spherical functions. Let

$$V = \{v \in R^n : [v, v] = 0, v_2 = 1\}, \quad V^\pm = V \cap \{v_1 \gtrless 0\}, \quad \tilde{\eta} = \eta_2 v.$$

Continuous series. To the generalized functions $\theta_p^\pm(\eta) = (\eta_2)_\pm^\sigma$ there correspond (see §1) the spherical functions

$$\Phi_\rho^\pm(x) = \int_Y [x^0, \tilde{\eta}]_\pm^\sigma [x, \tilde{\eta}]_\pm^{\tilde{\sigma}} d\eta = \int_{V^\pm} [x, v]_\pm^{\tilde{\sigma}} dv \quad \left(dv = \frac{dv_3 \cdots dv_n}{|v_1|} \right). \quad (1)$$

Discrete series. From $\theta_l(\eta)$ we obtain (see §1) the spherical functions

$$\Psi_l(x) = p_{nl} \int_Y [x_0, \tilde{\eta}]_+^l \beta_l([x, \tilde{\eta}]) d\eta = p_{nl} \int_{V^+} \beta_l([x, v]) dv, \quad (2)$$

where $p_{nl} = 2^{-\tau} \pi^{(4-n)/2} / \Gamma(l+1)$. (For $n = 3$, Ψ_l is the sum of two spherical functions.) The integrals (1), (2) are understood in the sense of the regularized value, see §5.

5. Regularization of integrals. Consider the integral

$$R_\lambda(x) = \int_{V^+} [x, v]_+^\lambda dv,$$

where λ is complex, $x \in X$. If $|x_2| > 1$, then it converges absolutely for values of λ lying in a certain domain, and is analytically continued to the points σ of interest to us from (1). If $|x_2| < 1$, then $R_\lambda(x)$ diverges for all λ . However, then the integral

$$R_{\lambda, \mu}(x) = \int_{V^+} [x, v]_+^\lambda |v_1|^\mu dv$$

converges absolutely for $\operatorname{Re}(\lambda + \mu) < 3 - n$, $\operatorname{Re} \lambda > -1$, and is analytically continued to the point $\lambda = \sigma$, $\mu = 0$. We proceed analogously with the integrals (2) and (3).

6. Expression of the spherical functions in terms of Legendre functions.

Continuous series. Let $\nu = (n - 3)/2$, $E(\rho) = \frac{2}{\pi} (2\pi)^\nu |\Gamma(\sigma + 1)|^2$. Then

$$\frac{\Phi_\rho^+(x)}{E(\rho)} = \begin{cases} -e^{-i\nu\pi} \sin \sigma\pi (x_2^2 - 1)^{-\nu/2} Q_{\nu+\sigma}^\nu(x_2), & \text{for } x_2 > 1, x_1 > 0, \\ -e^{-i\pi\nu} \sin \bar{\sigma}\pi (x_2^2 - 1)^{-\nu/2} Q_{-\nu-\bar{\sigma}-1}^\nu(x_2), & \text{for } x_2 > 1, x_1 < 0, \\ \frac{\pi}{2} (1 - x_2^2)^{-\nu/2} P_{\nu+\sigma}^\nu(-x_2), & \text{for } |x_2| < 1, \\ 0, & \text{for } x_2 < -1, x_1 > 0, \\ \pi \cos \pi\nu (x_2^2 - 1)^{-\nu/2} P_{\nu+\sigma}^\nu(-x_2), & \text{for } x_2 < -1, x_1 < 0. \end{cases}$$

The expressions for $\Phi_\rho^-(x)$ are obtained by replacing x_1 by $-x_1$.

Discrete series. If we denote

$$f(z) = (z^2 - 1)^{-\nu/2} Q_{l+\nu}^\nu(z),$$

then

$$\Psi_l(x) = 2^{\nu-l} \gamma_l \Gamma^{-1}(l + n - 2) [f(x_2 + i0) - (-1)^n f(x_2 - i0)].$$

All spherical functions are locally integrable.

7. Expansion of the δ -function in spherical functions. Let $\delta(x)$ denote the δ -function on X , concentrated at x^0 . For the expansion of $\delta(x)$ in spherical functions we use the method of M. Riesz, as in $(1, 2)$. Take the generalized function of one variable a

$$s(a, \lambda) = \frac{1}{2^{2\lambda} \Gamma(\lambda + \frac{1}{2})} |a - 1|^{2\lambda} a_+^{-\lambda-\nu}, \quad \nu = (n - 3)/2,$$

and put

$$S(x, \lambda) = \int_Y s([x, v], \lambda) dv, \quad x \in X. \quad (3)$$

The integral is understood in the sense of the regularization of Sect. 5. It is computed explicitly:

$$S(x, \lambda) = \{\cos \pi \lambda \cdot (1 - x_2)_-^\lambda + (1 - x_2)_+^\lambda\} S_1(x) S_2(\lambda), \quad (4)$$

where $S_1(x) = (x_2 + 1)^{-\nu}$ for odd n ; $S_1(x) = (x_2 + 1)_+^{-\nu}$ for even n ;

$$S_2(\lambda) = \pi^{-1/2-\nu} 2^{-\lambda+\nu+1} \Gamma(-\nu - \lambda).$$

Let $K(\lambda, \rho)$ be such that

$$s(a, \lambda) = \int_{-\infty}^{\infty} a_+^{(2-n+i\rho)/2} K(\lambda, \rho) d\rho.$$

Then, denoting $\Phi_\rho = \Phi_\rho^+ + \Phi_\rho^-$, we obtain from (1), (2)

$$S(x, \lambda) = \int_{-\infty}^{\infty} K(\lambda, \rho) \overline{\Phi_\rho(x)} d\rho. \quad (5)$$

Put $\lambda = -\nu - 1$. Using the results of (7), Ch. III, § 2, item 2, and also the relation $\Phi_\rho = \Phi_{-\rho}$, we obtain from (4) and (5)

$$\varepsilon_n \delta(x) = \frac{1}{2} \pi^{-\nu-2} \Gamma(\nu + 1) L(x) + \int_{-\infty}^{\infty} \omega(\rho) \overline{\Phi_\rho(x)} d\rho,$$

where

$$L(x) = (x_2 - 1)^{-1} (1 - x_2^2)^{-\nu} \quad \text{for odd } n; \quad L(x) = (x_2 - 1)^{-1} (1 - x_2^2)_+^{-\nu} \quad \text{for even } n;$$

$\varepsilon_n = \frac{1}{2}$ or 1, respectively, and

$$2^{2n-3} \pi^{n-1} \omega(\rho) = \begin{cases} \rho \tanh \frac{\pi \rho}{2} \cdot \prod_{k=1}^{\nu} [(2k-1)^2 + \rho^2], & n \text{ odd}, \\ \prod_{k=0}^{[\nu]} (4k^2 + \rho^2), & n \text{ even}. \end{cases}$$

We expand $L(x)$ in $\Psi_l(x)$, for which purpose we use the expressions of Ψ_l in terms of Legendre functions. Finally:

$$\delta(x) = \sum_{l=0}^{\infty} a_l \overline{\Psi_l(x)} + \int_{-\infty}^{\infty} \omega(\rho) \overline{\Phi_\rho(x)} d\rho,$$

where

$$a_l = \frac{1}{2} \pi^{-n/2-1} (2l + n - 2) \Gamma(l + n - 2) 4^{2-n-l}.$$

8. An analogue of the Plancherel formula. Let $C_c^\infty(X)$ be the space of compactly supported infinitely differentiable functions on X . We shall call the following functions on Y the Fourier components of a function $f \in C_c^\infty(X)$:

$$F_\rho^\pm(\eta) = \int f(x)[x, \eta]_\pm^\sigma dx, \quad \sigma = \frac{2-n+i\rho}{2},$$

$$F_l(\eta) = \int f(x)\beta_l([x, \eta]) dx.$$

For each ρ or l , these formulas define a mapping of $C_c^\infty(X)$ into D_τ , $\tau = i\rho$ or $\tau = 2-n-2l$, respectively. Under this, the shifts by g pass into T_g^τ .

From the expansion of $\delta(x)$ and the integral representations (1), (2), it follows that $f \in C_c^\infty(X)$ is reconstructed from its Fourier components as follows:

$$f(x) = \sum_{l=0}^{\infty} a_l B(F_l, \beta_l, ([x, \tilde{\eta}])) + \int_{-\infty}^{\infty} \omega(\rho) \{ (F_\rho^+, [x, \tilde{\eta}]_+^\sigma) + (F_\rho^- [x, \tilde{\eta}]_-^\sigma) \} d\rho.$$

The correspondence

$$f \rightarrow \{F_\rho^+, F_\rho^-, F_l\}$$

can be extended to

$$L^2(X) \rightarrow \{H_\rho, H_\rho, H_l\}.$$

Thus, the representation in $L^2(X)$ of the group G decomposes into representations of the continuous series with multiplicity 2 and representations of the discrete series with multiplicity 1. There is an analogue of the Plancherel formula

$$\int |f(x)|^2 dx = \sum_{l=0}^{\infty} a_l B(F_l, F_l) + \int_{-\infty}^{\infty} \omega(\rho) \{ (F_\rho^+, F_\rho^+) + (F_\rho^-, F_\rho^-) \} d\rho,$$

where a_l, ω_ρ are determined by the formulas preceding (7).

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Note: Figure translations are in progress. See original paper for figures.

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