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Abstract

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MATHEMATICS

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ON DEFECTIVE NONASYMPTOTIC VALUES OF MEROMORPHIC FUNCTIONS

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Let $f(z)$ be a function meromorphic in the finite z -plane. Throughout we shall use the standard notation of Nevanlinna theory ⁽¹⁾. If a is a Picard exceptional value of $f(z)$, then, by the well-known theorem of Iversen ^(1, p. 293), a is an asymptotic value for the function $f(z)$. It is also known ⁽²⁻⁵⁾ that in this theorem the requirement that a be a Picard exceptional value cannot be replaced by the weaker condition that a be a deficient value, even if one restricts oneself to entire functions. However, from results of Edrei and Fuchs ⁽⁶⁾ it follows that for entire functions of finite order, for which the value a has maximal deficiency $\delta(a) = 1$, a is necessarily an asymptotic value. The question naturally arises (see, for example, ⁽⁷⁾) whether this will also be true for meromorphic functions. Here we give a negative answer to this question. Namely, for any ρ , $1 < \rho \leq \infty$, we construct a meromorphic function of order ρ for which $\delta(\infty) = 1$ and ∞ is not an asymptotic value. Moreover, in our example the ratio $T(r, f)/r^\rho$ for $r > r_0$ is bounded above and below by positive constants if $1 < \rho < \infty$, and $\ln r = o(\ln T(r, f))$ if $\rho = \infty$, while $N(r, \infty) \sim Cr^\beta$, $\rho/(2\rho - 1) < \beta < 1$, $0 < C < \infty$. Thus ∞ is also a nonasymptotic Borel exceptional value*.

First let $1 < \rho < \infty$. Take on $[0, \infty)$ two continuously differentiable functions $l_j(r)$, $j = 1, 2$, possessing the following properties:

- A. $\alpha \leq l_j(r) \leq \rho$, where α is some number such that $\rho/(2\rho - 1) < \alpha < 1$.
- B. $|l'_j(r)| = O\{(r \ln r \ln_2 r \ln_3 r)^{-1}\}$, where $\ln_i r$ denotes the i -th iterated logarithm.
- C. There exist sequences of segments $[a_n^{(j)}, b_n^{(j)}]$, $n = 1, 2, \dots$; $j = 1, 2$, such that $a_n^{(j)} \rightarrow \infty$, $b_n^{(j)}/a_n^{(j)} \rightarrow \infty$ as $n \rightarrow \infty$, and $l_j(r) \equiv \alpha$ for $r \in [a_n^{(j)}, b_n^{(j)}]$.
- D. $l(r) = \max\{l_1(r), l_2(r)\} \equiv \rho$ for $0 \leq r < \infty$.

It is not difficult to verify that such functions $l_j(r)$ can be constructed.

Everywhere below we have $z = re^{i\varphi}$, $|\varphi| \leq \pi$. Consider the functions analytic in $|\varphi| < \pi$

$$\lambda_j(z) = z \int_0^\infty \frac{l_j(t) dt}{(t+z)^2}.$$

We shall show that in each angle $|\varphi| \leq \pi - \delta$, $\delta > 0$, the following holds uniformly (cf. ⁽¹⁰⁾, p. 53):

$$\lambda_j(z) = l_j(r) + o\{(\ln r)^{-1}\}. \quad (1)$$

* A Borel exceptional value may have zero deficiency. Apparently this was first noted by Valiron ⁽⁸⁾; see also ⁽⁹⁾, p. 271.

We note that for $|\varphi| \leq \pi - \delta$, $0 \leq t < \infty$, one has $|t + re^{i\varphi}| \geq \sin \delta/2(t + r)$. Hence

$$|\lambda_j(z) - l_j(r)| = \left| z \int_0^\infty \frac{l_j(t) - l_j(r)}{(t+z)^2} dt \right| \leq \frac{r}{\sin^2 \delta/2} \int_0^\infty \frac{|l_j(t) - l_j(r)|}{(t+r)^2} dt. \quad (2)$$

We split the last integral into three parts: over $[0, r(\ln r)^{-2}]$, over $[r(\ln r)^{-2}, r(\ln r)^2]$, and over $[r(\ln r)^2, \infty)$. In the first and third integrals we estimate $|l_j(t) - l_j(r)| \leq \rho$, and in the second

$$|l_j(t) - l_j(r)| = |\xi l'_j(\xi)| |\ln(t/r)| = O\{(\ln r \ln_3 r)^{-1}\},$$

where ξ lies between t and r (condition B was used, and $|\ln(t/r)| \leq 2 \ln_2 r$). Substituting in (2), we obtain (1). We note that (1) remains valid if condition A is replaced by the condition $l_j(r) = o(\ln r)$. We also note that for $|\varphi| \leq \pi - \delta$

$$z^{\lambda_j(z)} = (1 + o(1))z^{l_j(r)}. \quad (3)$$

Let θ be a constant such that $1 < \theta < 2\rho\alpha/(\rho + \alpha) < 2$. Denote by Γ_j the boundary of the domain

$$D_j = \{r > 1, |\varphi| < \pi\theta/2l_j(r)\},$$

which is taken so that D_j lies to the right of Γ_j . In the complement of $\overline{D_j}$ we define an analytic function $F_j(z)$ by an integral of Cauchy type (the convergence follows at once from (3)):

$$F_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{e^{\zeta^{\lambda_j(\zeta)}}}{\zeta - z} d\zeta, \quad z \in C\overline{D_j}, \quad (4)$$

and then analytically continue $F_j(z)$ to the whole finite z -plane, replacing in (4) Γ_j by the boundary of the domain $D_j \cap \{|z| > r_0\}$, $r_0 > 1$. Applying the usual arguments in these questions (cf., for example, ⁽¹⁰⁾, p. 54), for the entire function $F_j(z)$ we find

$$F_j(z) = \begin{cases} O(1), & \text{for } z \in CD_j, \\ \exp\{z^{\lambda_j(z)}\} + O(1), & \text{for } z \in D_j. \end{cases}$$

Denote by Δ_1 the domain D_1 , and by Δ_2 the domain symmetric to D_2 with respect to the imaginary axis. By our choice of θ , one has

$$\pi - \frac{\pi\theta}{2l_2(r)} > \frac{\pi\theta}{2l_1(r)},$$

and therefore Δ_1 and Δ_2 do not intersect. Let

$$\Delta_3 = C(\Delta_1 \cup \Delta_2).$$

Then for the function

$$\Psi(z) \equiv F_1(z) + F_2(-z)$$

we have

$$\Psi(z) = \begin{cases} \exp\{z^{\lambda_1(z)}\} + O(1), & \text{for } z \in \Delta_1, \\ \exp\{(-z)^{\lambda_2(-z)}\} + O(1), & \text{for } z \in \Delta_2, \\ O(1), & \text{for } z \in \Delta_3. \end{cases}$$

Let $\alpha < \beta < 1$ and

$$\Phi(z) = \prod_{n=1}^{\infty} \left[1 - \left(\frac{z}{n^{1/\beta}} \right)^2 \right].$$

Using the known formulas ⁽¹¹⁾, it is easy to obtain asymptotic expressions for $|\Phi(z)|$. If from $|z| < \infty$ we exclude certain disks with centers at $\pm n^{1/\beta}$ and radii r_n , $n = 1, 2, \dots$, such that

$$\sum_{k=1}^n r_k = o(n^{1/\beta}),$$

then in the remaining domain G , as $r \rightarrow \infty$, uniformly ($|\varphi| \leq \pi$),

$$\ln |\Phi(re^{i\varphi})| = (1 + o(1))A(\beta) \cos \beta(|\varphi| - \pi/2)r^\beta, \quad A(\beta) = \pi \operatorname{cosec} \pi\beta/2.$$

It is obvious that in the domain G the function $\Phi(z)$ tends uniformly to ∞ . We shall show that the function

$$f(z) = \Psi(z)/\Phi(z)$$

has the required properties. Indeed, taking into account (3) and the boundedness of the set $\Delta_3 \cap CG$,

we obtain

$$|f(z)| = \begin{cases} \exp\{(1+o(1))[r^{l_1(r)} \cos l_1(r)\varphi - A(\beta) \cos \beta(|\varphi| - \pi/2)r^\beta]\} + o(1), & \text{for } z \in \Delta_1 \cap G, \\ \exp\{(1+o(1))[r^{l_2(r)} \cos l_2(r)(\pi - |\varphi|) - A(\beta) \cos \beta(|\varphi| - \pi/2)r^\beta]\} \\ \quad + o(1), & \text{for } z \in \Delta_2 \cap G, \\ o(1), & \text{for } z \in \Delta_3. \end{cases} \quad (5)$$

It is easy to see that $n(r, \infty, f) = n(r, 0, \Phi) + O(1) \sim 2r^\beta$, $N(r, \infty, f) \sim (2/\beta)r^\beta$. Denote by I_1 the intersection of G with the positive real ray, $I_2 = (0, \infty) \setminus I_1$. Obviously, $\text{mes}\{(0, r) \cap I_2\} = o(r)$. Taking condition Γ into account, from (5) we obtain that, as $r \rightarrow \infty$, $r \in I_1$, one has $m(r, \infty, f) = (1+o(1))K(r)r^\rho$, where $(\pi\rho)^{-1} \leq K(r) \leq 2(\pi\rho)^{-1}$. Consequently, for $r \in I_1$ we have

$$T(r, f) = m(r, \infty) + N(r, \infty) = (1+o(1))K(r)r^\rho. \quad (6)$$

Since the function $T(r, f)$ is increasing, relation (6) is valid for all $r > 0$. Thus $\delta(\infty) = 1$, and ∞ is a Borel exceptional value.

We shall now show that ∞ is not an asymptotic value. For all sufficiently large n , in the segments $[a_n^{(j)}, b_n^{(j)}]$ there are points $r_n^{(j)} \in I_1$. On the arc $\{|z| = r_n^{(j)}, z \in \Delta_j\}$, $j = 1, 2$, by virtue of (5) and condition B, one has ($r = r_n^{(j)}$):

$$\begin{aligned} |f(re^{i\varphi})| &= \exp\{(1+o(1))[O(r^\alpha) - A(\beta) \cos \beta(|\varphi| - \pi/2)r^\beta]\} + o(1) \leq \\ &\leq \exp\{-(1+o(1))\pi \operatorname{ctg} \frac{\pi\beta}{2} r^\beta\} + o(1) = o(1). \end{aligned}$$

Since, moreover, $|f(re^{i\varphi})| = o(1)$ for $z \in \Delta_3$, it is clear that ∞ cannot be an asymptotic value.

In the case $\rho = \infty$ the example must be modified only slightly. Let $1/2 < \alpha < \beta < 1$, $1 < \theta < 2\alpha$. We choose the functions $l_j(r)$ so that they satisfy conditions B and C, and also the following conditions:

A'. $\alpha \leq l_j(r) < \infty$.

Γ' . $l(r) \rightarrow \infty$, $l(r) = o(\ln r)$ as $r \rightarrow \infty$.

D. If $l_1(r) < 2\theta\alpha/(2\alpha - \theta)$, then $l_2(r) > \theta\alpha$, and if $l_2(r) < 2\theta\alpha/(2\alpha - \theta)$, then $l_1(r) > \theta\alpha$.

After this the previous arguments go through almost without change.

Remark. As is known (¹¹, Ch. I, §§ 10, 11), an entire function of finite order can have a finite Borel exceptional value only when the order ρ is a natural number and the type is mean or maximal. If the order of $N(r, a)$ is less than the order of $T(r, f)$, then a is an asymptotic value (stronger results in this direction are also known (¹²)). But the type of $N(r, a)$ may be lower than the type of the entire function without a being an asymptotic value. We indicate corresponding examples. Let $\rho = 1$. Take for $f(z)$ the canonical product of genus 1 with zeros at points a_n on the real axis, $|a_n| = n$, the signs of a_n being chosen so that $\limsup_{n \rightarrow \infty} A_n = +\infty$,

$$\liminf_{n \rightarrow \infty} A_n = -\infty, \quad \text{where } A_n = \sum_{k=1}^n a_k^{-1}.$$

It is clear that $|f(iy)| \geq 1$, $-\infty < y < +\infty$. Using known results (¹¹, Ch. I, § 11), one can show that there exist semicircles $\{|z| = R_n, \operatorname{Re} z \geq 0\}$ and $\{|z| = R'_n, \operatorname{Re} z \leq 0\}$, $R_n \rightarrow \infty$, $R'_n \rightarrow \infty$, on which $|f(z)|$ tends uniformly to ∞ . Consequently, zero is not an asymptotic value. But $N(r, 0)$ has mean type, while the function $f(z)$ is of maximal type,

i.e., 0 is a Borel exceptional value. If we take $|a_n| = n \ln n$, then $N(r, 0)$ will be of minimal type, while the type of $f(z)$, as before, is maximal. If, with $|a_n| = n \ln n$, we choose the signs of a_n so that $\limsup_{n \rightarrow \infty} A_n = 1$, $\liminf_{n \rightarrow \infty} A_n = -1$, then the type of $N(r, 0)$ is minimal, the type of $T(r, f)$ is mean, and zero again will not be an asymptotic value. If ρ is an integer, $\rho > 1$, then we consider $f(z^\rho)$, where $f(z)$ are the functions indicated above.

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Note: Figure translations are in progress. See original paper for figures.

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