

SOME THEOREMS ON SOLUTIONS OF A PROBLEM IN METEOROLOGY

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Abstract

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MATHEMATICS

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SOME THEOREMS ON SOLUTIONS OF A PROBLEM IN METEOROLOGY

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Consider the following problem: the functions u, v, H satisfy the system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - lv + \frac{\partial H}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + lu + \frac{\partial H}{\partial y} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial}{\partial p} \frac{p^2}{\mu} \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial H}{\partial p} &= 0 \end{aligned} \quad (1)$$

for

$$t > 0, \quad 0 < p_0 < p < P;$$

the boundary conditions

$$\begin{aligned} \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial H}{\partial p} &= 0 \quad \text{for } p = p_0, \\ p \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \frac{\partial H}{\partial p} + \alpha \frac{\partial H}{\partial t} &= 0 \quad \text{for } p = P; \end{aligned} \quad (2)$$

the initial conditions

$$u(x, y, 0, p) = u^0(x, y, p), \quad v(x, y, 0, p) = v^0(x, y, p),$$

$$H(x, y, 0, p) = H^0(x, y, p). \quad (3)$$

Here l is an analytic function of x, y ; α is an analytic function of x, y, t , with $\alpha \geq \alpha_0 > 0$; μ is a continuously differentiable function of p , with $0 < \mu_0 \leq \mu \leq \mu_1$; $p_0, P, \alpha_0, \mu_0, \mu_1$ are constants.

Definition 1. We shall say that a function $f(x_1, \dots, x_n, p)$, analytic in $x_1, \dots, x_n$ in a neighborhood of the point $(x_1^0, \dots, x_n^0)$, admits a majorant uniform in p in the interval $p_0 < p < P$ if it expands in the series

$$f(x_1, \dots, x_n, p) = \sum_{k_i=0}^{\infty} f_{k_1 \dots k_n}(p) (x_1 - x_1^0)^{k_1} \dots (x_n - x_n^0)^{k_n}$$

and there exist constants $a_{k_1 \dots k_n}$ such that

$$|f_{k_1 \dots k_n}(p)| \leq a_{k_1 \dots k_n} \quad \text{for } p_0 < p < P,$$

and the series

$$\sum_{k_i=0}^{\infty} a_{k_1 \dots k_n} (x_1 - x_1^0)^{k_1} \dots (x_n - x_n^0)^{k_n}$$

converges in some neighborhood of the point $(x_1^0, \dots, x_n^0)$.

Theorem 1. If the initial data u^0, v^0, H^0 are functions analytic in x, y in a neighborhood of the point (x_0, y_0) , such that $u^0, v^0, H^0, \partial u^0 / \partial p, \partial v^0 / \partial p, \partial H^0 / \partial p, \partial^2 H^0 / \partial p^2$ admit majorants uniform in p in the interval $p_0 < p < P$, then there exists a unique-

...solution of problem (1)–(3), analytic in x, y, t in a neighborhood of the point $(x_0, y_0, 0)$, such that $u, v, H, \partial u / \partial p, \partial v / \partial p, \partial H / \partial p, \partial^2 H / \partial p^2$ admit, in a neighborhood of this point, majorants uniform in p in the interval $p_0 < p < P$.

We shall outline the proof. In addition to the functions u, v, H , introduce the functions $u_p = \partial u / \partial p$, $v_p = \partial v / \partial p$, $H_p = \partial H / \partial p$, and $H_{pp} = \partial^2 H / \partial p^2$. The system (1) can be solved with respect to the time derivatives of the functions u, v, H_{pp} . Differentiating the first two equations with respect to p , we obtain equations solved with respect to $\partial u_p / \partial t$ and $\partial v_p / \partial t$. Using the third equation of system (1) and the boundary conditions (2), we obtain equations for H_p and H .

$$\frac{\partial H_p}{\partial t} = -u \frac{\partial H_p}{\partial x} - v \frac{\partial H_p}{\partial y} + \frac{\mu}{p^2} \int_{p_0}^p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp,$$

$$\frac{\partial H}{\partial t} = \int_p^P \left[-u \frac{\partial H_p}{\partial x} - v \frac{\partial H_p}{\partial y} + \frac{\mu}{p^2} \int_{p_0}^p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp \right] dp - \frac{\mu(P)}{\alpha P} \int_{p_0}^P \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp.$$

The obtained system of equations (denote it by (1')) makes it possible to compute uniquely all derivatives with respect to x, y, t as functions of p at the point $(x_0, y_0, 0)$. The problem is therefore reduced to proving the convergence of formal series. Note that system (1') contains derivatives with respect to x and y only of first order and contains no derivatives with respect to p . The integration with respect to p present in two equations can increase the coefficients of the series in x, y, t by no more than a bounded factor. Using this circumstance, it is not difficult to construct a Cauchy–Kovalevskaya system that will be a majorant with respect to system (1'). The solutions of this system will not depend on p , whence the assertion of Theorem 1 follows regarding majorants uniform in p .

In the subsequent considerations we weaken the smoothness requirements on α to the requirement: α and $\partial\alpha/\partial t$ are continuous for $t \geq 0$. Let (u_1, v_1, H_1) and (u_2, v_2, H_2) be two different solutions of problem (1)–(3) such that all derivatives entering into the equations of system (1) are continuous. The functions u, v, H , defined by the formulas

$$u = (u_1 - u_2)e^{-\lambda t}, \quad v = (v_1 - v_2)e^{-\lambda t}, \quad H = (H_1 - H_2)e^{-\lambda t}, \quad (4)$$

where λ is a positive constant, satisfy the system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u_1 \frac{\partial u}{\partial x} + v_1 \frac{\partial u}{\partial y} + \lambda u + lv + \frac{\partial u_2}{\partial x} u + \frac{\partial u_2}{\partial y} v + \frac{\partial H}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + u_1 \frac{\partial v}{\partial x} + v_1 \frac{\partial v}{\partial y} + \lambda v - lu + \frac{\partial v_2}{\partial x} u + \frac{\partial v_2}{\partial y} v + \frac{\partial H}{\partial y} &= 0, \end{aligned} \quad (5)$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial p} \frac{p^2}{\mu} \left[\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + \lambda \right) \frac{\partial H}{\partial p} + \frac{\partial^2 H_2}{\partial x \partial p} u + \frac{\partial^2 H_2}{\partial y \partial p} v \right] = 0,$$

with boundary conditions

$$\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + \lambda \right) \frac{\partial H}{\partial p} + \frac{\partial^2 H_2}{\partial x \partial p} u + \frac{\partial^2 H_2}{\partial y \partial p} v = 0 \quad \text{for } p = p_0,$$

$$p \left[\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + v_1 \frac{\partial}{\partial y} + \lambda \right) \frac{\partial H}{\partial p} + \frac{\partial^2 H_2}{\partial x \partial p} u + \frac{\partial^2 H_2}{\partial y \partial p} v \right] + \alpha \frac{\partial H}{\partial [[unclear : denominator]]} + \lambda H = 0 \quad \text{for } p = P. \quad (6)$$

and the initial conditions

$$\begin{aligned} u(x, y, 0, p) &= u_1^0 - u_2^0, \\ v(x, y, 0, p) &= v_1^0 - v_2^0, \\ H(x, y, 0, p) &= H_1^0 - H_2^0. \end{aligned} \quad (7)$$

Let Ω be some volume with piecewise smooth boundary S in the space (x, y, t) . Multiplying the system of equations (5) scalarly by the vector (u, v, H) and, using the boundary conditions (6), integrating the result over Ω and over p from p_0 to P , we obtain the identity

$$\int_{\Omega} \left(\frac{\partial Q}{\partial t} + \frac{\partial R}{\partial x} + \frac{\partial L}{\partial y} + K \right) dx dy dt = 0, \quad (8)$$

where

$$Q = \frac{1}{2} \int_{p_0}^P \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] dp + \frac{\alpha P}{2\mu(P)} H^2(P), \quad (9)$$

$$R = \int_{p_0}^P \left\{ Hu + \frac{u_1}{2} \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] \right\} dp, \quad (10)$$

$$L = \int_{p_0}^P \left\{ Hv + \frac{v_1}{2} \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] \right\} dp, \quad (11)$$

$$\begin{aligned} K &= \left(\lambda \alpha - \frac{1}{2} \frac{\partial \alpha}{\partial t} \right) \frac{P}{\mu(P)} H^2(P) + \int_{p_0}^P \left\{ \frac{\partial u_2}{\partial x} u^2 + \frac{\partial v_2}{\partial y} v^2 \right. \\ &\quad + \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) uv + \frac{p^2}{\mu} \frac{\partial H}{\partial p} \left(u \frac{\partial^2 H_2}{\partial x \partial p} + v \frac{\partial^2 H_2}{\partial y \partial p} \right) \\ &\quad \left. + \left[\lambda - \frac{1}{2} \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) \right] \left[u^2 + v^2 + \frac{p^2}{\mu} \left(\frac{\partial H}{\partial p} \right)^2 \right] \right\} dp. \end{aligned} \quad (12)$$

Since, by assumption, the derivatives of u_1, v_1, H_1 and u_2, v_2, H_2 entering K are continuous, they are bounded in Ω , and by choosing a sufficiently large λ one can obtain the inequality

$$\int_{\Omega} K dx dy dt \geq 0. \quad (13)$$

Let Ω be a truncated cone with bases in the planes $t = 0$ and $t = t_1$; S_0 the lower base of the cone Ω ; S_1 the upper base; S_2 the lateral surface of Ω . Suppose that on S_2 the equality

$$\operatorname{tg} nt = \frac{1}{C + V_0}, \quad (14)$$

is satisfied, where n is the outward normal to S_2 ,

$$v_0 = \max_{S_2} \sqrt{u_1^2 + v_1^2}, \quad (15)$$

$$p_0 \leq p \leq P$$

$$C = \max\{\sqrt{4\mu_1}, \sqrt{2\mu(P)/\alpha_0}\}. \quad (16)$$

If (u_1, v_1, H_1) and (u_2, v_2, H_2) coincide on S_0 , then, taking (13) into account and applying the Ostrogradsky-Gauss formula, instead of (8) we shall have

$$\int_{S_1} Q \, dx \, dy \leq - \int_{S_2} (Q \cos nt + R \cos nx + L \cos ny) \, dS. \quad (17)$$

It is verified directly, taking into account (14), (15), and (16), that

$$\int_{S_2} (Q \cos nt + R \cos nx + h \cos ny) \, dS \geq 0. \quad (18)$$

From (18), (17), and (9) it follows that $u = v = H = 0$ on S_1 .

Thus we have proved

Theorem 2. Problem (1)–(3) has at most one smooth solution.

Let Ω be a right cylinder with bases in the planes $t = 0$ and $t = t_1$; S_0 the lower base of Ω ; S_1 the upper base of Ω ; S_2 the lateral surface of Ω .

Definition 2. We shall say that a point (x, y, t, p) , $(x, y, t) \in S_2$, is an inflow point of the solution (u_1, v_1, H_1) if at this point the inequality

$$u_1 \cos nx + v_2 \cos ny < 0, \quad (19)$$

is satisfied, where $\mathbf{n}$ is the outward normal to S_2 .

Theorem 3. If (u_1, v_1, H_1) , (u_2, v_2, H_2) are two solutions of problem (1)–(3), coinciding on S_0 and at the inflow points, and $H_1 = H_2$ on S_2 , then $(u_1, v_1, H_1) \equiv (u_2, v_2, H_2)$ in Ω .

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Note: Figure translations are in progress. See original paper for figures.

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