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Abstract

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MATHEMATICS

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RATIONAL POINTS OF A CLASS OF ALGEBRAIC CURVES

(Presented by Academician I. M. Vinogradov on 4 III 1966)

The results set forth in the present note concern the problem of finding all rational points of a class of curves of higher genus.

Let K be a field of algebraic numbers of degree n over the field of rational numbers. Consider the curve

$$\Gamma : F(x, y) = 0$$

of genus > 1 , admitting over the field K r linearly independent mappings φ_i ($i = 1, 2, \dots, r$) onto one and the same elliptic curve

$$\Gamma_1 : \Phi(u, v) = 0.$$

The problem is to find all points of the curve Γ rational over the given field K , under certain restrictions imposed on the curves Γ and Γ_1 .

It is known that the mappings of the curve Γ onto the curve Γ_1 and the points of the curve Γ_1 over K form groups G and U with a finite number of generators. To each point a of the curve Γ there correspond r points $\varphi_i(a)$ ($i = 1, 2, \dots, r$) of the curve Γ_1 .

Two methods are indicated for solving the problem posed.

The first method consists in investigating the points $\varphi_i(a)$ ($i = 1, 2, \dots, r$) for linear dependence; in doing so the Tate height of points and the properties of height expounded in the book ⁽¹⁾ are used in an essential way. In this direction the following results have been obtained.

Let $g(\varphi)$ be the degree of the mapping φ ; let e be a function taking only the values ± 1 ; and let H_a be the Tate height of the point a . Then the following theorems hold.

Theorem 1. *If*

$$g\left(\sum_{i=1}^r e_i \varphi_i\right) > g\left(\sum_{\substack{i=1 \\ i \neq k}}^r e_i \varphi_i\right) \quad (k = 1, 2, \dots, r)$$

for all values e_i , then the points $\varphi_i(a)$ ($i = 1, 2, \dots, r$), $a \in \Gamma$, starting from some C when $H_a > C$, are linearly independent.

Note that in Theorem 1 the mappings φ_i ($i = 1, 2, \dots, r$) are not assumed to be linearly independent; this already follows from the theorem itself.

Theorem 2. *If the mappings φ_i ($i = 1, 2, \dots, r$) are linearly independent, then the points of the curve Γ_1 , $\varphi_i(a)$ ($i = 1, 2, \dots, r$), $a \in \Gamma$, starting from some C when $H_a > C$, are also linearly independent.*

Theorem 3 (main). *If the curve Γ admits over the field K r linearly independent mappings onto one and the same elliptic curve Γ_1 of rank $< r$, then the curve Γ has a finite number of points rational over this field.*

Note that if the mappings of the curve Γ onto the curve Γ_1 are known, then, when the conditions of Theorem 3 are satisfied, the constant C can be effectively found, and consequently all points of the curve Γ rational over the given field can be found.

Particular cases of the proved theorem are the following examples, considered at the end of the paper ⁽²⁾:

Example 1. The curve $x^4 + y^4 = A$ has a finite number of points rational over the field K , $A \in K$, if the curve $u^3 + v^2 = A$ has rank ≤ 1 over K .

Example 2. The curve $x^4 + y^4 = A$ has a finite number of points rational over the field K , $A \in K$, if the curve $u^4 + v^2 = A$ has rank ≤ 1 over K .

With respect to the curves considered in Examples 1 and 2, we prove the following additional theorems.

Theorem 4. *The curve $x^6 + y^6 = A$ has a finite number of points rational over the field $R(1)$, if the rank of one of the curves $\Gamma_1 : u^3 + v^2 = A$, $\Gamma_2 : Au^3 - 1 = v^2$, $\Gamma_3 : u^3 + 1 = Av^2$ over $R(1)$ does not exceed 1, or if the ranks of any two of these curves do not exceed 2.*

Theorem 5. *If the rank of the curve $u^4 + v^2 = A$ over the field $R(1)$ does not exceed 2, and the rank of the curve $u^4 + 1 = Av^2 - 1$, then the curve $x^4 + y^4 = A$ has a finite number of rational points.*

In the proof of these theorems one uses the fact that the curves $x^4 + y^4 = A$, $x^6 + y^6 = A$ admit, respectively over the fields $R(\sqrt[4]{-A})$, $R(\sqrt[6]{-A})$, a greater number of mappings onto one and the same elliptic curve than over the field $R(1)$.

In the second method, one studies the distribution of the points $\varphi_i(a)$ ($i = 1, 2, \dots, r$) among the elements of the factor group $U/2U$, without using the Tate height. In this method, curves of genus > 1 for which the order of the automorphism group is equal to 2 are considered in greater detail. The following theorems are proved.

Theorem 6. *If the rank of one of the curves $\Gamma_1, \Gamma_2, \Gamma_3$ over the field $R(\varepsilon)$, $\varepsilon^2 + \varepsilon + 1 = 0$, does not exceed 2, then the curve $x^6 + y^6 = A$ has no points in this field, except for the cases: $A = 1$, $(x, y) = (\varepsilon_1, 0), (0, \varepsilon_2)$; $A = 2$, $(x, y) = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1^6 = \varepsilon_2^6 = 1$.*

Theorem 7. *If the rank of the curve $u^4 + v^2 = A$ over the field $R(\sqrt{-1})$ does not exceed 2, then the curve $x^4 + y^4 = A$ has no points in this field, except for the cases: $A = 1$, $(x, y) = (\varepsilon_1, 0), (0, \varepsilon_2)$; $A = 2$, $(x, y) = (\varepsilon_1, \varepsilon_2)$; $\varepsilon_1^4 = \varepsilon_2^4 = 1$.*

Theorem 8. *If the rank of the curve $(A^2 - 4)u^4 + 2A(A + 2)u^2 + A^2 - 4 = v^2$ over the field $R(1)$ does not exceed 2, then the curve $x^4 + y^4 + 1 = A(x^2y^2 + x^2 + y^2)$ has no rational points, except for the cases: $A = (2a^4 + 1)/(a^4 + 2a^2)$, $(x, y) = (\pm a, \pm a)$; $A = (a^4 + 2)/(2a^2 + 1)$, $(x, y) = (\pm a, \pm 1), (\pm 1, \pm a)$.*

In the proof of Theorems 6, 7, 8 it is established: first, that none of the points $\varphi_i(a)$ ($i = 1, 2, \dots, r$) of the corresponding elliptic curve can fall into the class $2U$; second, that no two points among $\varphi_i(a)$ can fall into one and the same class of the factor group $U/2U$.

Under the condition that $K = R(1)$, we put forward the following conjecture for the curves of Theorems 6, 7, 8: if the points $\varphi_i(a), \varphi_i(b)$ ($i = 1, 2, \dots, r$), $r \geq 2$, are distributed entirely in two classes of the factor group $U/2U$, then $a = b$. From the validity of this hypothesis the following results would follow:

1. The curves $x^4 + y^4 = A$, $x^4 + y^4 + 1 = A(x^2y^2 + x^2 + y^2)$ have in the field $R(1)$, respectively, no more than $2^{r_1-1}(2^{r_1} - 1)$, $2^{r_2-1}(2^{r_2} - 1)$ points, where r_1, r_2 are the ranks of the curves $u^4 + v^2 = A$, $(A^2 - 4)u^4 + 2A(A + 2)u^2 + A^2 - 4 = v^2$ over this field.
2. The curve $x^6 + y^6 = A$ has in the field $R(1)$ no more than $2^{s-1}(2^s - 1)$ points, where $s = \min(r_1, r_2, r_3)$, and r_1, r_2, r_3 are the ranks of the curves $u^3 + v^2 = A$, $Au^3 - 1 = v^2$, $u^3 + 1 = Av^2$ over this field.

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2. Yu. I. Manin, *Izv. AN SSSR, ser. matem.*, No. 6, 1363 (1964).

Note: Figure translations are in progress. See original paper for figures.

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