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Abstract

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MATHEMATICS

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OPERATIONAL CALCULUS AND PALEY-WIENER TYPE THEOREMS FOR A SEMISIMPLE COMPLEX LIE GROUP

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As is known, on every semisimple complex Lie group (and even on a much broader class of groups) there exists an analogue of the ordinary harmonic Fourier analysis. The corresponding L^2 -theory (an analogue of the Plancherel formula, the inversion formula) was first constructed in the well-known works of I. M. Gel' fand and M. A. Naimark ⁽¹⁾, and also of Harish-Chandra ⁽²⁾. However, as has already been noted more than once, for many questions, chiefly those connected with the theory of infinite-dimensional representations, a fundamentally important role is played by subtler theorems of harmonic analysis of the Paley-Wiener type. In this direction the Lorentz group, which has rank 1, was investigated first. Following the method of M. A. Naimark ⁽³⁾, the author obtained an analogue of the Paley-Wiener theorem for the class of finite functions on the Lorentz group ^(4,5). At approximately the same time I. M. Gel' fand obtained an analogous theorem for the class of functions "rapidly decreasing at infinity" ⁽⁶⁾. It should be noted (see ⁽⁷⁾) that Gel' fand's theorem can be obtained as an easy consequence of the author's theorem*. Subsequently the theorem of I. M. Gel' fand was generalized by him jointly with M. I. Graev ⁽⁸⁾ to an arbitrary semisimple complex group G , whereas the author's attempts to obtain a similar generalization of his theorem had until now been unsuccessful. At the present time the author has succeeded in applying a new simple method and in completely solving the problem.

In this note we shall consider the following classes of functions on a semisimple complex connected group G : E is the space of all generalized functions with compact supports; K is the space of all square-integrable functions with compact supports; U is the space of generalized functions concentrated at the identity element $e \in G$; X is the space of all finite infinitely differentiable functions; X_∞ is the space of all rapidly decreasing (i.e. decreasing faster than any power $\|g\|^{-n}$) infinitely differentiable functions. All these spaces are group algebras, i.e. are closed with respect to convolution on the group G .

Recall that the Fourier transform of a function $x(g)$ on the group G is defined by means of the formula

$$T(\alpha) = \int x(g) e(\alpha, g) dg, \quad (1)$$

* We also note that if one speaks of applications to representation theory, then it is necessary to estimate the convergence of integrals of the form $T_x = \int x(g) T_g dg$, where $x(g)$ is a function on the group and T_g is the representation operator. Rapidly decreasing functions are, generally speaking, unsuitable for representations in Banach spaces, which, as is known, grow exponentially, whereas finite functions are applicable to representations in arbitrary locally convex spaces.

where $e(a, g)$ is the operator of the elementary representation $e(a)$ of the group G (for the definition and the basic properties of elementary representations, see (9)). To generalized functions such a transformation can be extended in the usual way by means of the Plancherel formula. If the function $x(g)$ is finite, then its Fourier transform is meaningful for an arbitrary complex signature a and has the property of **double symmetry**:

$$WT(a_s) = T(a)W, \quad ST(a) = T(a')S, \quad (2)$$

which follows from the analogous property of elementary representations $e(a)$ (see (9)). If the function $x(g)$ is continuous, then the operator $T(a)$ is a Hilbert-Schmidt operator in the space $L_2(\mathfrak{U})$

$$\|T(a)\|_2 < \infty$$

($\|T\|_2 = \{\text{Sp}(T^*T)\}^{1/2}$ is the Hilbert-Schmidt norm).

Finally, if the function $x(g)$ is infinitely differentiable, then the conditions

$$\|\rho^n \Delta^{n_1} T(a) \Delta^{n_2}\|_2 < \infty, \quad n, n_1, n_2 = 0, 1, 2, \dots, \quad (3)$$

are satisfied, where Δ is the Laplace-Beltrami operator on the group $\mathfrak{U}$ and $\rho^n = (\rho_1 \rho_2 \dots \rho_r)^n$ (here $\rho = (\rho_1, \rho_2, \dots, \rho_r)$ is the spectral parameter of the signature a).

For each space Φ , composed of functions $x(g)$, basic or generalized, we shall denote by $\tilde{\Phi}$ the image of Φ under the Fourier transform (the dual space). By a theorem of Paley-Wiener type for the space Φ we understand an independent

description of the space $\tilde{\Phi}$, i.e. a description of the conditions on the operator-valued function $T(a)$ that are necessary and sufficient for its inverse Fourier transform to belong to the class Φ .

We now pass to the statement of the results.

I. Operational calculus on the group G . Our method consists in a preliminary study of the algebra U , which may be interpreted as the algebra of differential operators generated by left or right translations on the group G .^{*} Just as in ordinary Fourier analysis differential operators (with constant coefficients) are transformed comparatively simply, it is natural to expect that the algebra U can also be readily described. In fact, instead of the algebra U we shall study all its possible “finite-dimensional projections,” i.e. matrix elements (which is natural from the point of view of the Fourier transform on the group G), and agree to understand such a transformation as an “operational calculus” for the group G .

Let us first introduce the necessary definitions connected with finite-dimensional projections. Let Φ be a linear space composed of functions $x(g)$; put

$$\Phi_{ij} = e_i \Phi e_j,$$

where $\{e_i\}$, $i = 0, 1, 2, \dots$, is a system of minimal projectors generated by left and right translations by elements of the maximal compact subgroup $\mathfrak{U} \subset G$. If $x \in \Phi_{ij}$, then, as is known, the Fourier transform of the function x is a one-dimensional operator T_x with the sole nonzero matrix element

$$t_{ij} = (T_x \xi_j, \xi_i),$$

where $\{\xi_i\}$, $i = 0, 1, 2, \dots$, is the “canonical basis” in the space $\mathcal{H} = L_2(\mathfrak{U})$.^{**} We agree on such a numbering that the projector e_0 corresponds to the identity representation of the group $\mathfrak{U}$.

If Φ is one of the algebras E, K, X, X_∞ , then Φ is closed under multiplication by the projectors e_i , i.e. in this case $\Phi_{ij} \subset \Phi$. Moreover, obviously,

$$\Phi_{ij} \Phi_{jl} \subset \Phi_{il},$$

^{*} In an unpublished work of L. Schwartz it is proved that the algebra U is isomorphic to the universal enveloping algebra for the Lie algebra of the group G (see (10)).

^{**} Multiplication by the projector e_i may be regarded as convolution with a certain measure concentrated on the set $\mathfrak{U}$; in this case the Fourier transform of the measure e_i is the projection operator in $\mathcal{H}$ onto the direction ξ_i .

and, in particular, each set Φ_{ij} is a subalgebra with involution in the algebra Φ . Moreover, every set of the form

$$\Phi_N = \sum_{i,j \in N} \Phi_{ij},$$

where N is an arbitrary finite set of indices i, j , is a subalgebra. At the same time the entire algebra Φ can be obtained as the closure of an expanding system of subalgebras Φ_N .

If $\Phi = U$, then $\Phi_{ij} \not\subset \Phi$; however, in this case the situation is considerably facilitated by the circumstance that the (nonclosed) linear span of the basis vectors ξ_i is invariant with respect to the operations of the algebra U . Moreover, although this is far from obvious from the definition, the sets U_{ii}, U_N are still algebras. The study of the algebras U_N still gives detailed information about the structure of the algebra U .

Since each projector e_i is connected with a certain irreducible representation of the group $\mathfrak{U}$, we can assign to each index i the uniquely corresponding highest weight $k = k(i)$ of this irreducible representation. We fix an index i and first consider the algebra $A = A_{ii}$, dual to U_{ii} with respect to the Fourier transform.

Basic lemma. *The algebra A contains an ideal I , which can be represented in the form*

$$I = \sum_{l < k} A_{ij} A_{ji},$$

where A_{ij} is the set dual to U_{ij} , and the index $l = l(j)$ runs through all highest weights subordinate to the highest weight $k = k(i)$.

The factor algebra $F = A/I$ is isomorphic to the algebra of all polynomials

$$f(\rho) = f(\rho_1, \rho_2, \dots, \rho_r)$$

in r complex variables (where r is the rank of the group G), having the symmetry property

$$f(s\rho) = f(\rho),$$

where s is an arbitrary element of the Weyl group preserving the vector k ($sk = k$).

Sketch of proof. The assertion of the lemma is comparatively easy to verify in the case $i = 0$. In this case, by our convention, $k = 0$, and hence $I = (0)$. Consequently, $F = A$, and A consists of the polynomials $f(\rho)$ symmetric with

respect to the entire Weyl group. A proof of this fact can be obtained in various ways.* Thus the algebra U_{00} is described. Next we use the relations

$$U_{00}U_{0i} \subset U_{0i}, \quad U_{i0} = (U_{0i})^*, \quad U_{i0}U_{0i} \subset U_{ii}$$

(* denotes involution) and apply the methods for studying matrix elements of infinitesimal operators developed in (9). In this way the first part of the lemma is obtained. The second part, concerning the structure of the factor algebra F , also requires a rather complicated proof.

In a similar manner an arbitrary algebra A_N , dual to U_N , can be described, and this may be regarded as the principal result of the operational calculus for the group G :

Theorem 1. *The algebra A_N consists of all possible functions $T(\alpha)$, concentrated on a finite-dimensional subspace** $\mathcal{H}_N \subset \mathcal{H}$ and possessing the following two properties:*

1. $T(\alpha)$ is a polynomial in the variables $\rho = (\rho_1, \rho_2, \dots, \rho_r)$, where ρ is the spectral parameter of the signature α .
2. $T(\alpha)$ satisfies the symmetry conditions following from (2).

From Theorem 1 there follows a number of interesting consequences concerning representations of the algebra U , but we shall not dwell on them now.

* For example, from results of F. A. Berezin on the structure of Laplace operators of the group G .

** $\mathcal{H}_N$ is the range of the projector in $\mathcal{H}$ generated by the measure $e_N = \sum_{i \in N} e_i$.

Remark. Since $U_{ij} \notin U$, instead of the algebra U it would perhaps be natural to consider the algebra Ω , consisting of all generalized functions on the group G concentrated on the set $\mathfrak{U}$. This algebra is already invariant with respect to e_i . However, in fact

$$\Omega_N = U_N$$

for all indices N , and we obtain nothing new in the study of finite-dimensional projections. Nevertheless, it is interesting to note that the algebra Ω is the maximal group algebra all of whose projections are transformed into polynomials under the Fourier transform.

II. The Paley–Wiener theorem for finite functions. The application of operational calculus to the study of the algebra X is based on the obvious fact that X is closed under multiplication by elements of U , i.e., is a two-sided U -module. Hence it follows, in particular, that we are entitled to use relations of the form

$$U_{ij}X_{jl} \subset X_{il}$$

for studying the structure of the algebra X . However, independently it is still necessary to study matrix elements analogous to $f(\rho)$ from the main lemma, which is achieved with the aid of the Plancherel formula. Having thus studied the algebras X_N and passing to the limit with respect to N , we find a description of the algebra $\widetilde{X}$ dual to X :

Theorem 2. *The algebra $\widetilde{X}$ consists of all operator functions $T(\alpha)$, acting in the space $\mathcal{H}$ and satisfying the conditions:*

1. $T(\alpha)$ is a Hilbert–Schmidt operator with the restrictions (3).
2. $T(\alpha)$ is an entire analytic function of growth order not exceeding one and of finite type, with estimates of the form

$$\|\rho^n T(\alpha)\| \leq C_n e^{a|\operatorname{Re} \rho|}, \quad n = 0, 1, 2, \dots, \quad (*)$$

where $\rho^n = (\rho_1 \rho_2 \dots \rho_r)^n$ and the constant C_n depends on the function $T(\alpha)$.

3. $T(\alpha)$ is subject to the relations of double symmetry (2).

Moreover, fulfillment of the conditions (*) is equivalent to the fact that the corresponding function $x(g)$ vanishes outside the compact set $\ln \|g\| \leq a$.

Remark. We have given a global formulation of the result; however, for applications to representation theory the study of the finite-dimensional projections X_N themselves is no less important. In particular, the “Naimark subalgebras” $Y = X_{ii}$ play a special role. The structure of these objects is not difficult to describe by analogy with the main lemma and Theorem 1; in particular, the algebra dual to Y is analogous to the algebra A from the main lemma, with polynomials replaced by entire functions of exponential growth.

III. Other theorems of Paley–Wiener type. Since the Paley–Wiener theorem has been obtained for finite functions, we can, as indicated in paper (7), derive analogous results for the algebra E (an analogue of the Paley–Wiener–Schwartz theorem), the algebra K (an analogue of the classical Paley–Wiener theorem), and the algebra X_∞ (the Gelfand–Graev theorem). All proofs in our case remain almost unchanged.

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- * Here $\|g\|$ is the “unitary norm” in the group G , which satisfies the condition

$$\|u_1 g u_2\| = \|g\|$$

for any $u_1, u_2 \in \mathfrak{U}$.

Note: Figure translations are in progress. See original paper for figures.

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