

ON THE THEORY OF HOTELLING' S TEST

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Abstract

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MATHEMATICS

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ON THE THEORY OF HOTELLING' S TEST

Let $X_1, \dots, X_N$ be independent normal vectors of a p -dimensional Euclidean space, having common mean $\xi = EX_i$ (a p -dimensional column vector) and common correlation matrix

$$\Sigma = E'[(X_i - \xi)(X_i - \xi)^\tau],$$

$i = 1, \dots, N$; here τ denotes transposition. Put

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N X_i;$$

$$S = \sum_{i=1}^N (X_i - \bar{X})(X_i - \bar{X})^\tau.$$

Consider the problem of testing the statistical hypothesis $H_0 : \xi = 0$ against the (composite) alternative $H_\delta : N\xi^\tau \Sigma^{-1} = \delta$, where δ is an arbitrary fixed positive number. Obviously, this problem may be interpreted, in a certain sense, as testing for the presence, in a given Gaussian noise with zero mean, of a signal of given energy. For its solution one usually uses Hotelling' s T^2 -test (see ⁽¹⁾), rejecting the hypothesis H_0 when

$$T^2 = N(N-1)\bar{X}^\tau S^{-1}\bar{X} > T_0^2,$$

where T_0^2 is a constant determined by the chosen level α . We note that for a constant value of δ the power of the T^2 -test is also constant (see ⁽¹⁾).

The statistical properties of the T^2 -test were studied by J. Simaika ⁽²⁾, C. Stein ^(3,4), N. Giri, J. Kiefer, and C. Stein ^(5,6). The test has a simple structure, but its properties are mysterious. Up to now not a single nontrivial case is known for which the admissibility of the T^2 -test has been proved. A weaker property here, minimaxity, was justified in ⁽⁶⁾ only for $p = 2$, $N = 3$. In this very interesting work the authors, relying on the Hunt-Stein theorem (see ⁽⁷⁾), reduce the problem of minimaxity of the T^2 -test to the investigation of a Fredholm integral equation of the first kind, which for $p = 2$, $N = 3$ is transformed into an "overdetermined" linear differential equation of the first order; the explicit form of the solution of the latter makes it possible to prove

the minimaxity of the T^2 -test in this special case. In the general case the Fredholm integral equation has the form

$$\int_{\Gamma_1} \exp \left\{ \sum_{j=1}^p t_j \sum_{i>j} \frac{\beta_i}{2} \right\} \cdot \prod_{i=1}^p \Phi \left(\frac{N-i+1}{2}, \frac{1}{2}, \frac{t_i \beta_i}{2} \right) d\lambda^*(\beta_1, \dots, \beta_p) = \Phi \left(\frac{N}{2}, \frac{p}{2}, \frac{\gamma}{2} \right),$$

where Γ_1 denotes the $(p-1)$ -dimensional simplex

$$\{(\beta_1, \dots, \beta_p) : \beta_i \geq 0, \sum_{i=1}^p \beta_i = 1\}$$

and Φ is the confluent hypergeometric function. The equation must hold identically in $t_1, \dots, t_p$, satisfying the condition

$$\sum_{i=1}^p t_i = \gamma,$$

where γ is any fixed positive number, and the unknown function λ^* must be a probability measure on Γ_1 . The minimaxity of the T^2 -test is equivalent to the existence of such a λ^* .

In the present note we continue the investigation of N. Giri, J. Kiefer, and C. Stein and prove the minimaxity of the T^2 -test under the alternative

H_δ for $p = 2$, $N = 4$. A slight complication of our arguments also gives the construction of an overdetermined boundary-value problem with a linear differential operator in the general case ($p = 2$).

For $p = 2$, $N = 4$ the problem consists in showing that on the interval $0 < \beta < 1$ there exists a probability density $\lambda(\beta)$ for which

$$\int_0^1 e^{t(1-\beta)/2} \Phi \left(2, \frac{1}{2}, \frac{t\beta}{2} \right) \Phi \left(\frac{3}{2}, \frac{1}{2}, \frac{(\gamma-t)(1-\beta)}{2} \right) \lambda(\beta) d\beta = \Phi \left(2, 1, \frac{\gamma}{2} \right). \quad (1)$$

With a view to solving this problem, let us consider the equation

$$\begin{aligned} \int_0^1 e^{t(1-\beta)/2} \Phi \left(2, \frac{1}{2}, \frac{t\beta}{2} \right) \Phi \left(\frac{3}{2}, \frac{1}{2}, \frac{(\gamma-t)(1-\beta)}{2} \right) \lambda(\beta) d\beta = \\ = \int_0^1 \Phi \left(\frac{3}{2}, \frac{1}{2}, \frac{\gamma(1-\beta)}{2} \right) \lambda(\beta) d\beta. \end{aligned} \quad (2)$$

Using the fact that

$$\Phi \left(\frac{3}{2}, \frac{1}{2}, \frac{x}{2} \right) = (1+x)e^{x/2},$$

we rewrite it in the form

$$\begin{aligned} \int_0^1 e^{-\gamma\beta/2} \Phi\left(2, \frac{1}{2}, \frac{t\beta}{2}\right) [1 + \gamma - \gamma\beta - t(1 - \beta)] \lambda(\beta) d\beta = \\ = \int_0^1 e^{-\gamma\beta/2} (1 + \gamma - \gamma\beta) \lambda(\beta) d\beta. \end{aligned}$$

The left-hand side of the last equality can be expanded in a power series in t and the system obtained

$$\int_0^1 e^{-\gamma\beta/2} [(r - 2r^2)\beta^{r-1} + (1 + \gamma + \gamma r + 2r^2)\beta^r - (\gamma + \gamma r)\beta^{r+1}] \lambda(\beta) d\beta = 0,$$

$$r = 1, 2, \dots \quad (3)$$

Let

$$e^{\gamma\beta/2} f(\beta) = \sum_{r=1}^{\infty} a_{r-1} \beta^{r-1}$$

be an arbitrary function representable by a series whose radius of convergence is greater than one. Multiply the r -th equation (3) by a_{r-1} and sum over r from 1 to ∞ . After some calculations we find

$$\int_0^1 \left[(2\beta^3 - 2\beta^2) f'' + (-5\beta + 6\beta^2 - \gamma\beta^2 + \gamma\beta^3) f' + \left(-1 + 3\beta - \frac{\gamma\beta}{2} + \gamma\beta^2 \right) f \right] \lambda(\beta) d\beta = \int_0^1 L(f) \lambda(\beta) d\beta = 0 \quad (4)$$

Choose a small $\varepsilon > 0$ and apply Green's formula to the differential form $L(f)$:

$$\int_{\varepsilon}^{1-\varepsilon} L(f) \lambda(\beta) d\beta = \int_{\varepsilon}^{1-\varepsilon} L^*(\lambda) f(\beta) d\beta + \mathcal{L}[f, \lambda] \Big|_{\varepsilon}^{1-\varepsilon},$$

where the adjoint form is

$$L^*(\lambda) = 2\beta^2(\beta - 1)\lambda'' + \beta(-3 + 6\beta + \gamma\beta - \gamma\beta^2)\lambda' + \left(3\beta + \frac{3\gamma\beta}{2} - 2\gamma\beta^2 \right) \lambda$$

and the bilinear form is

$$\mathcal{L}[f, \lambda] = 2\beta^2(1 - \beta)(f\lambda' - f'\lambda) - [\beta + \gamma\beta^2(1 - \beta)]f\lambda.$$

From Frobenius theory (8) it follows that the equation $L^*(\lambda) = 0$ has two fundamental solutions in the interval $(0, 1)$,

$$\lambda_{01} = \sum_{k=0}^{\infty} a_k \beta^k, \quad \lambda_{02} = \beta^{-1/2} \sum_{k=0}^{\infty} c_k \beta^k \quad (5)$$

and two fundamental solutions

$$\lambda_{11} = \sum_{k=0}^{\infty} g_k (1 - \beta)^k, \quad \lambda_{12} = (1 - \beta)^{-1/2} \sum_{k=0}^{\infty} h_k (1 - \beta)^k, \quad (6)$$

where $a_0, c_0, g_0, h_0 \neq 0$. Hence any solution of the equation $L^*(\lambda) = 0$ is integrable on $[0, 1]$. We assert, moreover, that there exists a solution of this equation for which

$$\lim_{\varepsilon \rightarrow 0} \mathcal{L}[f, \lambda] \Big|_{\varepsilon}^{1-\varepsilon} = 0. \quad (7)$$

Indeed, considering linear combinations of the solutions (5), and then (6), one can easily find that (7) holds if and only if $\lambda = \lambda_{12}$.

The preceding arguments allow us to conclude that $\lambda = \lambda_{12}$ is a solution of equation (4) and hence of equation (2). We now prove that $\lambda = \lambda_{12}$ does not vanish in the interval $(0, 1)$.

The equation $L^*(\lambda) = 0$, after the substitution $\beta = 1 - x$, $v(x) = \frac{1}{h_0} \sqrt{x} \lambda(1 - x)$, becomes the equation

$$2x(x - 1)v'' + (-1 + 4x - \gamma x + \gamma x^2)v' + (3/2 + 3\gamma x/2)v = 0, \quad (8)$$

and the subsequent substitution $u = -v'/v$ gives us the Riccati equation

$$u' = u^2 - \frac{1 - 4x + \gamma x - \gamma x^2}{2x(1 - x)}u - \frac{3 + 3\gamma x}{4x(1 - x)}. \quad (9)$$

Recall that

$$v(x) = \sum_{k=0}^{\infty} \frac{h_k}{h_0} x^k, \quad |x| < 1.$$

Suppose that the function $v(x)$ vanishes somewhere in the interval $(0, 1)$. Since $v(0) = 1$, among the roots of the equation $v(x) = 0$ there will be a smallest one. Denote it by x_0 . Then the function u is analytic in the disk $|x| < x_0$. From (8), $v'(0) = 3/2$, i.e. $u(0) = -3/2$. Therefore $u(x)$ cannot vanish in the interval $(0, x_0)$, since at the point where this occurs for the first time one would have to have $u'(x) \geq 0$, but equation (9) shows that at this point $u'(x) < 0$. The negativity of the function $u(x)$ leads to unbounded decrease as one approaches x_0 from the left ($v'(x_0) \neq 0$, since $v \not\equiv 0$), which again contradicts (9). All this means that $\lambda = \lambda_{12}$ has no zeros in the interval $(0, 1)$.

Let us return to equation (2), which, as we have seen, is satisfied by the function $\lambda = \lambda_{12} \neq 0$. Using the method of Laplace transformations, it is not difficult to obtain the relation

$$\begin{aligned} & \int_0^1 x^{b-1} (1-x)^{c-b-1} e^{qx} \Phi(a, b, px) \Phi(a-b, c-b, q(1-x)) dx \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \Phi(a, c, p+q), \quad 0 < b < c. \end{aligned} \quad (10)$$

First putting $t = \gamma\tau$ in (2), we multiply both sides of this equality by $\tau^{-1/2}(1-\tau)^{-1/2}$ and integrate with respect to τ from 0 to 1. We find the integral with respect to τ , taking in (10) $a = 2$, $b = 1/2$, $c = 1$, $p = \gamma\beta/2$, $q = \gamma(1-\beta)/2$. As a result it turns out that

$$\int_0^1 \Phi\left(2, 1, \frac{\gamma}{2}\right) \lambda(\beta) d\beta = \int_0^1 \Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{\gamma(1-\beta)}{2}\right) \lambda(\beta) d\beta.$$

To obtain (1), it suffices to use the homogeneity of the written relation and to normalize the function $\lambda = \lambda_{12}$ accordingly.

In conclusion we make a small remark. It can be shown that from the minimaxity of the T^2 -test under any alternative H_δ , $\delta > 0$, there follows the minimaxity of the T^2 -test under the more natural alternative $H'_\delta: N\xi'\Sigma^{-1}\xi \geq \delta$. The proof is by contradiction, if one takes into account that, as we have already mentioned, for fixed δ the power of the T^2 -test is constant. This remark is also substantiated by I. V. Romanovskii.

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