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V. F. Dem'yanov

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Abstract

Full Text

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MATHEMATICS

V. F. DEM' YANOV

ON A NECESSARY CONDITION FOR AN EXTREMUM IN CONTROL PROBLEMS WITH AFTEREFFECT

(Presented by Academician L. S. Pontryagin on 14 V 1965)

Let U be a convex, bounded, weakly closed class of summable real r -dimensional vector-functions on $[0, T]$ ($T > 0$ fixed). The class U is called the **class of controls**, and a vector-function

$$u(t) = (u^1(t), \dots, u^r(t)) \in U$$

is called a **controlling vector-function** or a **control**. For example, as U one may consider classes of summable r -dimensional vector-functions on $[0, T]$ satisfying on $[0, T]$ one of the following constraints:

$$|u^i(t)| \leq \alpha_i(t); \quad \alpha_i(t) \geq 0; \quad i = 1, \dots, r; \quad t \in [0, T]; \quad (1)$$

$$u^*(t)N(t)u(t) \leq \beta(t); \quad \beta(t) \geq 0; \quad t \in [0, T]; \quad (2)$$

$$\int_0^T u^{i2}(t) dt \leq C_i; \quad 0 < C_i < \infty; \quad i = 1, \dots, r; \quad (3)$$

$$\int_0^T u^*(t)N(t)u(t) dt \leq C; \quad 0 < C < \infty, \quad (4)$$

where $\alpha_i(t)$ ($i = 1, \dots, r$), $\beta(t)$ are piecewise continuous functions bounded on $[0, T]$; $N(t)$ is an $r \times r$ matrix, positive definite on $[0, T]$, with piecewise continuous entries bounded on $[0, T]$; the asterisk denotes transposition.

We consider the problem of finding the minimum on U of a certain functional $I(u)$, described below. A general problem of this kind was considered, in particular, in (1).

On $[0, T]$ a system of n differential equations with variable delay is given:

$$dx(t)/dt = \dot{x}(t) = f(x(t), x(t - h_1(t)), u(t), t); \quad (5)$$

$$x(0) = x_0(t) \quad \text{for } t \in [-h_1(0), 0]. \quad (6)$$

The function $v(t) = t - h_1(t)$ is a strictly increasing continuously differentiable real function on $[0, T]$,

$$0 < h_1(t) < \infty \quad \text{for } t \in [0, T]; \quad \min_{t \in [0, T]} h_1(t) > 0.$$

Then there exists an inverse function $t = r_1(v)$, which is also a strictly increasing continuously differentiable real function on $[-h_1(0), T - h_1(T)]$; $u \in U$, $f(x, y, u, t) = (f^1, \dots, f^n)$ is an n -dimensional real vector-function, continuous in x^i, y^j, u^k, t and continuously differentiable in x^i, y^j, u^k ($i, j = 1, \dots, n$; $k = 1, \dots, r$) in the domain of admissible values x^i, y^j, u^k , determined by the class of controls U , the system (5), (6), and the initial vector-function $x_0(t)$, given and continuous on $[-h_1(0), 0]$.

By $x(t, u) = (x^1, \dots, x^n)$ we denote the solution of system (5) with initial conditions (6).

A functional is given

$$I(u) = \int_0^T g(x(t, u), x(t - h_2(t), u), u(t), t) dt, \quad (7)$$

where $\nu(t) \equiv t - h_2(t)$ is a strictly increasing continuously differentiable real function on $[0, T]$; $0 \leq h_2(t) < \infty$ for $t \in [0, T]$. Let $t = r_2(\nu)$ be the function inverse to $\nu(t)$ (it also is strictly increasing and continuously differentiable on $[-h_2(0), T - h_2(T)]$); $g(x, y, u, t)$ is a scalar real function continuous in x^i, y^j, u^k, t and continuously differentiable in x^i, y^j, u^k ($i, j = 1, \dots, n$; $k = 1, \dots, r$) in the domain of admissible values of x^i, y^j, u^k and for $t \in [0, T]$.

We assume that if $h_2(0) > h_1(0)$, then $x(t)$ is prescribed and continuous on $[-h_2(0), -h_1(0)]$.

From the class of controls U it is required to find a control $u \in U$ such that

$$I(u) = \min_{v \in U} I(v). \quad (8)$$

Such a $u \in U$ is called an optimal control.

Theorem. *In order that the control $u \in U$ give the functional (7) its minimal value on U , it is necessary, and in the case of convexity of $I(u)$ also sufficient, that*

$$\min_{v \in U} \int_0^T \sum_{i=1}^r \left[\left(\frac{\partial f(\tau)}{\partial u^i} \right)^* \psi(\tau) + \frac{\partial g(\tau)}{\partial u^i} \right] (v^i(\tau) - u^i(\tau)) d\tau = 0, \quad (9)$$

where $\psi(\tau)$ is an n -dimensional vector-function satisfying the system of differential equations

$$\dot{\psi}(\tau) = \begin{cases} -\left(\frac{\partial f(\tau)}{\partial x}\right)^* \psi(\tau) - \left(\frac{\partial f(r_1(\tau))}{\partial y}\right)^* \psi(r_1(\tau)) \dot{r}_1(\tau) - C(\tau), & \text{for } \tau \in [0, T - h_1(T)], \\ -\left(\frac{\partial f(\tau)}{\partial x}\right)^* \psi(\tau) - C(\tau), & \text{for } \tau \in [T - h_1(T), T], \end{cases} \quad (10)$$

$$\psi(T) = 0; \quad (11)$$

$$f(\tau) = f(x(\tau, u), x(\tau - h_1(\tau), u), u(\tau), \tau);$$

$$g(\tau) = g(x(\tau, u), x(\tau - h_2(\tau), u), u(\tau), \tau);$$

$$\partial g / \partial x = (\partial g / \partial x^1, \dots, \partial g / \partial x^n); \quad \partial g / \partial y = (\partial g / \partial y^1, \dots, \partial g / \partial y^n);$$

$$\partial f / \partial x = \begin{pmatrix} \partial f^1 / \partial x^1 & \dots & \partial f^1 / \partial x^n \\ \dots & \dots & \dots \\ \partial f^n / \partial x^1 & \dots & \partial f^n / \partial x^n \end{pmatrix}; \quad \partial f / \partial y = \begin{pmatrix} \partial f^1 / \partial y^1 & \dots & \partial f^1 / \partial y^n \\ \dots & \dots & \dots \\ \partial f^n / \partial y^1 & \dots & \partial f^n / \partial y^n \end{pmatrix};$$

$$C(\tau) = \begin{cases} \frac{\partial g(\tau)}{\partial x} + \frac{\partial g(r_2(\tau))}{\partial y} \dot{r}_2(\tau), & \text{for } \tau \in [0, T - h_2(T)], \\ \frac{\partial g(\tau)}{\partial x}, & \text{for } \tau \in [T - h_2(T), T]. \end{cases}$$

The proof of this theorem is carried out using the results obtained in (1).

If delay is absent both in system (5) and in functional (7) (i.e. $h_1(t) \equiv h_2(t) \equiv 0$), then the necessary condition (9), as is easy to see, is preserved, and the system for $\psi(\tau)$ instead of (10) will have the form

$$\dot{\psi}(\tau) = -(\partial f(\tau) / \partial x)^* \psi(\tau) - \partial g(\tau) / \partial x, \quad (12)$$

$$\psi(T) = 0. \quad (13)$$

It is clear that in this case, for classes of controls with constraints of the form (1) and (2), condition (9) is a “linearization” of L. S. Pontryagin’s “maximum principle”⁽²⁾. Verifying the fulfillment of condition (9) is simpler than verifying the fulfillment of the “maximum principle.” Moreover, the necessary condition

(9) is valid for classes of controls U of a more general form than the “maximum principle” (in particular, for classes of controls with constraints of integral type (3), (4)).

Similarly, for the case of a constant delay

$$h_1(t) \equiv h_2(t) \equiv h = \text{const}$$

a necessary condition has been obtained which is (for classes of controls of the form (1), (2)) a linearization of the “maximum principle” ((²), pp. 236–250). For the case of a variable delay (and classes of controls of the form (1), (2)), a necessary condition analogous to the “maximum principle” was obtained by Yu. F. Kazarinov (³).

A necessary condition analogous to (9) can be obtained for systems of integro-differential equations.

To find a control satisfying condition (9), one may apply one of the methods of successive approximations proposed in (¹).

Leningrad State University
named after A. A. Zhdanov

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