

ON SOLUTIONS OF NONLINEAR EQUATIONS OF THE FIRST ORDER

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Abstract

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MATHEMATICS

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ON SOLUTIONS OF NONLINEAR EQUATIONS OF THE FIRST ORDER

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In the paper ⁽¹⁾ a definition is given of a generalized solution of the Cauchy problem in the whole space for the equation

$$u_t + f(t, x, u, u_x) = 0, \quad (1)$$

$$x = (x_1, \dots, x_n), \quad u_x = (\partial u / \partial x_1, \dots, \partial u / \partial x_n), \quad u_t = \partial u / \partial t$$

with the initial condition

$$u(0, x) = u_0(x) \quad (2)$$

and theorems on the existence and uniqueness of such a solution are proved under the assumption that the function $f(t, x, u, v)$, $v = (v_1, \dots, v_n)$, is convex with respect to the last vector argument v . The present article continues this work.

In the first part, some methods are considered for constructing a generalized solution of problem (1), (2); alongside the proof of convergence of approximate solutions, error estimates for these solutions are obtained here. In the second part, some qualitative properties of generalized solutions of equation (1) are given, analogous to the properties of solutions of elliptic and parabolic equations in well-known Liouville theorems. As in ⁽¹⁾, for simplicity of formulation we restrict ourselves here to consideration of equation (1) with the function $f \equiv f(v)$ (this is the basic case).

Let $f(v)$ be twice continuously differentiable, $f_i = \partial f / \partial v_i$, $f_{ij} = \partial^2 f / \partial v_i \partial v_j$, and for any real $(\xi_1, \dots, \xi_n)$, $\sum_{i=1}^n \xi_i^2 = 1$, when

$$|v| = \left(\sum_{i=1}^n v_i^2 \right)^{1/2} \leq K$$

$$\sum_{i,j=1}^n f_{ij}(v) \xi_i \xi_j \geq \lambda(K) > 0. \quad (3)$$

Suppose that the function $u_0(x)$ has continuous derivatives of first and second order, and moreover $|(u_0)_x| \leq K_0$.

1. Some methods for constructing generalized solutions of problem (1), (2). We shall carry out the basic considerations using the example of the “vanishing viscosity” method. Let $u^\varepsilon(t, x)$ be the solution of the Cauchy problem in the strip $\Pi_T = \{(t, x) : 0 \leq t \leq T, x \in R_n(x)\}$ for the parabolic equation

$$u_t + f(u_x) = \varepsilon \Delta u, \quad 0 < \varepsilon < 1, \quad (4)$$

with the initial condition (2); here Δ is the Laplace operator with respect to the variables $x_1, \dots, x_n$.^{*} Denote by S_N the ball $\{x : |x| \leq N\}$, and by Q_N the cylinder $S_N \times [0, T]$. By C we shall denote arbitrary constants depending only on N, n, T and on the functions $u_0(x)$ and $\lambda(K)$.

^{*} The proof of convergence of the sequence $\{u^\varepsilon\}$ as $\varepsilon \rightarrow +0$, based on the uniqueness of the generalized solution of problem (1), (2), is given in (1).

It follows readily from the maximum principle that $|u_x^\varepsilon| \leq C$.

Theorem 1. Let the functions $u^{\varepsilon_1}(t, x)$ and $u^{\varepsilon_2}(t, x)$ be solutions of problem (4), (2) for $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$, respectively. Then

$$\max_{Q_N} |u^{\varepsilon_1}(t, x) - u^{\varepsilon_2}(t, x)| \leq C(\varepsilon_1 + \varepsilon_2)^{1/(n+1)}, \quad (5)$$

$$\int_{S_N} |\text{grad}(u^{\varepsilon_1} - u^{\varepsilon_2})| dx \leq C(\varepsilon_1 + \varepsilon_2)^{1/2}. \quad (6)$$

The proof of the theorem is based on the following two lemmas.

Lemma 1. For any direction l in the space $R_n(x)$,

$$\max_{Q_N} \frac{\partial^2 u^\varepsilon(t, x)}{\partial l^2} \leq C. \quad (7)$$

An analogous lemma was proved in [1].

Lemma 2. For $0 \leq t_0 \leq T$,

$$\int_{S_N} |u^{\varepsilon_1}(t_0, x) - u^{\varepsilon_2}(t_0, x)| dx \leq C(\varepsilon_1 + \varepsilon_2). \quad (8)$$

For the proof, multiply the equation satisfied by the function $u^{\varepsilon_1} - u^{\varepsilon_2}$ by a smooth function $g(t, x)$, finite with respect to x , and integrate over the layer Π_{t_0} . Integrating by parts, we find that

$$\begin{aligned} \int g(u^{\varepsilon_1} - u^{\varepsilon_2})|_{t=t_0} dx - \iint_{\Pi_{t_0}} (u^{\varepsilon_1} - u^{\varepsilon_2}) A(g) dx dt = \\ = \iint_{\Pi_{t_0}} g(\varepsilon_1 \Delta u^{\varepsilon_1} - \varepsilon_2 \Delta u^{\varepsilon_2}) dx dt, \end{aligned} \quad (9)$$

where

$$\begin{aligned} A(g) \equiv g_t + \sum_{i=1}^n \int_0^1 f_i(\dots) d\tau g_{x_i} + \int_0^1 \sum_{i,j=1}^n f_{ij}(\dots) (\tau u_{x_i}^{\varepsilon_1} + (1-\tau)u_{x_i}^{\varepsilon_2}) x_j d\tau g, \\ (\dots) = (\tau u_{x_i}^{\varepsilon_1} + (1-\tau)u_{x_i}^{\varepsilon_2}). \end{aligned}$$

Let $\eta(r) \in C^\infty([0, +\infty))$, $0 \leq \eta(r) \leq 1$, $\eta(r) = 0$ for $r \geq N+1$, and $\eta(r) = 1$ for $0 \leq r \leq N$. In (9), take as $g(t, x)$ the solution of the following Cauchy problem:

$$A(g) = 0, \quad g(t_0, x) = \eta(|x|)z_m(x),$$

where $z_m(x) \in C^\infty(R_n(x))$, $|z_m| \leq 1$, and $z_m \rightarrow \text{sgn}(u^{\varepsilon_1} - u^{\varepsilon_2})|_{t=t_0}$ as $m \rightarrow \infty$ in $L_1(S_{N+1})$. It is not difficult to see that $g(t, x) = 0$ for $|x| \geq C$; using condition (3) and Lemma 1, as in [1], we establish that the coefficient of the function g in the operator A is bounded above, whence it follows that $|g| \leq C$. Taking (7) into account, we find that the absolute value of the right-hand side in identity (9) does not exceed $C(\varepsilon_1 + \varepsilon_2)$. Letting $m \rightarrow \infty$ in (9), we complete the proof of Lemma 2.

To obtain estimate (5), we apply to the function $w = u^{\varepsilon_1} - u^{\varepsilon_2}$ the inequality

$$\max_{S_N} |w| \leq c(n) \left[I_0^{1/(n+1)} I_1^{n/(n+1)} + \frac{1}{\text{mes } S_N} I_0 \right],$$

where

$$I_0 = \int_{S_N} |w| dx, \quad I_1 = \max_{S_N} |w_x|;$$

estimate (6) follows from the inequality (see [2])

$$\| |w_x| \|_{L_1(S_N)} \leq c(n, N) \left[\left(\sum_{i,j=1}^n \|w_{x_i x_j}\|_{L_1(S_N)} \|w\|_{L_1(S_N)} \right)^{1/2} + \|w\|_{L_1(S_N)} \right],$$

if one takes into account that, by virtue of (7),

$$\|w_{x_i x_j}\|_{L_1(S_N)} \leq C,$$

and from equation (4)

$$\|w_t\|_{L_1(S_N)} \leq C(\|w_x\|_{L_1(S_N)} + \varepsilon_1 + \varepsilon_2).$$

Corollary. From Theorem 1 and inequality (7) it follows that $\{u^\varepsilon(t, x)\}$, as $\varepsilon \rightarrow +0$, converges to the generalized solution $u(t, x)$ of problem (1), (2), and moreover

$$\max |u - u^\varepsilon| \leq C\varepsilon^{1/(n+1)}$$

and

$$\|\text{grad}(u - u^\varepsilon)\|_{L_1(S_N)} \leq C\varepsilon^{1/2}.$$

Analogous estimates with right-hand sides of the form $C(\varepsilon \ln 1/\varepsilon)^\nu$ are also valid in the case of an arbitrary initial function $u_0(x)$ satisfying a Lipschitz condition.

Let us note that in the proof of Theorem 1 only the estimates

$$|u_x^\varepsilon| \leq C, \quad \|u_t^\varepsilon + f(u_x^\varepsilon)\|_{L_1(S_N)} \leq C\varepsilon,$$

inequality (7), and the condition $u^\varepsilon(0, x) = u_0(x)$ were used. This circumstance makes it possible to consider analogously other methods for constructing a solution of problem (1), (2), provided the approximate solutions obtained possess the indicated properties,* where the first derivatives here may be understood in the generalized sense, and inequality (7) may be replaced by the estimate

$$\frac{\Delta^2 u^\varepsilon}{\Delta x^2} \equiv \frac{u^\varepsilon(t, x + \Delta x) + u^\varepsilon(t, x - \Delta x) - 2u^\varepsilon(t, x)}{|\Delta x|^2} \leq C, \quad (10)$$

where $\Delta x = (\Delta x_1, \dots, \Delta x_n) \neq 0$.

As an example, let us consider, in the case $n = 1$, the finite-difference method with the P. Lax scheme (see (4)): $u_{\tau h}(k\tau, mh) = u_m^k$, where τ and h are certain positive numbers, k and m are integers, $k \geq 0$,

$$u_m^{k+1} = \frac{u_{m+1}^k + u_{m-1}^k}{2} - \tau f\left(\frac{u_{m+1}^k - u_{m-1}^k}{2h}\right), \quad u_m^0 = u_0(mh).$$

Assuming that

$$h/\tau \geq \sup_x |f'(u'_0(x))|,$$

we establish the estimates

$$|u_{m+1}^k - u_m^k| \leq Ch, \quad u_{m+j}^k + u_{m-j}^k - 2u_m^k \leq C(jh)^2,$$

where C does not depend on τ and h . From the values u_m^k it is not difficult to construct a piecewise-smooth function $u_h^\tau(t, x)$ in the half-plane $t \geq 0$, satisfying the conditions:

$$|u_h^\tau(0, x) - u_0(x)| \leq Ch \leq C_1(h^2/\tau),$$

$$|(u_h^\tau)_x| \leq C, \quad \Delta^2 u_h^\tau / \Delta x^2 \leq C$$

and

$$\|(u_h^\tau)_t + f((u_h^\tau)_x)\|_{L_1(-N, N)} \leq C(h^2/\tau).$$

The method proposed above makes it possible to obtain for u_h^τ estimates analogous to (5), (6), where $\varepsilon = (h^2/\tau)^{**}$

2. Theorem of Liouville Type

In (1) the property was established of stabilization, as $t \rightarrow +\infty$, of generalized solutions of problem (1), (2), defined in the half-space

$$\Pi_+ = \{(t, x) : t > 0, x \in R_n\},$$

to $\inf u_0(x)$. The following theorem concerns the properties of generalized solutions of the equation

$$\mathcal{L}(u) \equiv u_t + f(u_x) = 0$$

in the space $R_{n+1}(t, x)$ and in the half-space

$$\Pi_- = R_{n+1} \setminus \Pi_+.$$

Definition. A function $u(t, x)$ is called a generalized solution of the equation $\mathcal{L}(u) = 0$ in Π_- (or in $R_{n+1}(t, x)$) if $u(t, x)$ satisfies a Lipschitz condition in any bounded domain $\mathcal{D} \subset \Pi_-$ (or $\mathcal{D} \subset R_{n+1}(t, x)$), $\mathcal{L}(u) = 0$ almost everywhere in Π_- (or in $R_{n+1}(t, x)$),

$$\Delta^2 u / \Delta x^2 \leq C(t)$$

(see (10)), where the function $C(t)$ is bounded above on every interval $[-T, 0]$ (or on $[-T, \tilde{T}]$).

We may assume that $f(0) = 0$, since this can always be achieved by the change

$$\bar{u} = u + tf(0).$$

Theorem 2. 1) Let the function $u(t, x)$ be a generalized solut—

* See also (3), Theorem 3.

** The difference method considered admits a generalization to the case $n > 1$; for the corresponding approximate solutions, estimates (5), (6) have been established.

in Π_- , and $|u_x| \leq K_0$ almost everywhere in Π_- ; if $u(t, x) \geq |x|\mu(|x|) + K(t)$, $\mu(r) \leq 0$, $\lim_{r \rightarrow +\infty} \mu(r) = 0$, then $u(t, x) = \text{const}$ in Π_- .

2) Let $u(t, x)$ be a generalized solution in $R_{n+1}(t, x)$; if $|u_x| \leq K_0$ almost everywhere in R_{n+1} and $|\Delta^2 u / \Delta x^2| \leq K(t)$, where the function $K(t)$ is continuous, then $u(t, x)$ is linear and, consequently, representable in the form

$$u(t, x) = \text{const} + \sum_{i=1}^n v_i x_i - f(v)t.$$

The proof is based on the results of the work [1].

The following example shows that the fulfillment of the one-sided estimate (10) in case 2) is insufficient for the assertion of the theorem to be valid: the function $u = \min(x_1 - t, 0)$ is a generalized solution of the equation $u_t + (u_{x_1})^2 = 0$ in the plane (t, x_1) , satisfies condition (10), but is not linear. Moreover, since $u(t, x_1) \leq 0$, it follows from this example that in case 1) of Theorem 2 the lower estimate for the function $u(t, x)$ cannot be replaced by an upper estimate.

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Note: Figure translations are in progress. See original paper for figures.

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