

STABILITY OF MAGNETIC SURFACES

L. S. Solov'ev

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Abstract

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Physics

L. S. SOLOV' EV

STABILITY OF MAGNETIC SURFACES

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In the present work an approximate integral is obtained for the equations of the force lines of a magnetic field having a large longitudinal component, in an arbitrary geometry. With the aid of this integral, the problem of the stability of a configuration of toroidal magnetic surfaces with respect to small perturbations of the magnetic field is considered.

1. Integral of magnetic surfaces. The equations of the force lines of the magnetic field $\mathbf{B}$ in an arbitrary curvilinear coordinate system x^1, x^2, x^3 with metric $dl^2 = g_{ik}dx^i dx^k$ have the form

$$dx^1/dx^3 = B^1/B^3, \quad dx^2/dx^3 = B^2/B^3, \quad (1)$$

where B^i are the contravariant components of the vector $\mathbf{B}$. If the magnetic field has a large longitudinal component along some direction x^3 and, in comparison with it, small transverse components, then, provided the field is periodic in x^3 , there exist approximate (averaged over x^3) magnetic surfaces of such a field, depending on x^1 and x^2 .

We shall assume that g_{ik} are periodic functions of x^3 with the same period as $\mathbf{B}$. Multiply the numerators and denominators of the right-hand sides of equations (1) by $\sqrt{g}$, where $g = \text{Det } g_{ik}$, and represent $\sqrt{g}B^3$ as the sum of a constant and a variable (in x^3) part, $\sqrt{g}B^3 = \overline{\sqrt{g}B^3} + \widetilde{\sqrt{g}B^3}$. The straight bar above denotes averaging over the period of variation of x^3 at fixed x^1 and x^2 , and the wavy bar denotes the variable part, defined as the difference between the quantity itself and its mean value. Under the assumption that $\widetilde{\sqrt{g}B^3} \ll \overline{\sqrt{g}B^3}$, equations (1), to accuracy up to quadratic terms, may be written in the form

$$\frac{dx^k}{dx^3} = \frac{\sqrt{g} B^k}{\overline{\sqrt{g}B^3}} \left(1 - \frac{\widetilde{\sqrt{g}B^3}}{\overline{\sqrt{g}B^3}} \right) \equiv f^k(x^1, x^2, x^3).$$

According to the assumptions made, the right-hand sides f^k are small and periodic in x^3 . The approximate solution of these equations, obtained by the averaging method, is described by the formulas ⁽¹⁾

$$x^k = \bar{x}^k + \hat{f}^k(\bar{x}^1, \bar{x}^2, x^3) + \dots, \quad \frac{d\bar{x}^k}{dx^3} = \bar{f}^k + \frac{\partial \bar{f}^k}{\partial x^i} \hat{f}^i + \dots;$$

the operation $\hat{f}$ denotes the variable part of the indefinite integral with respect to x^3 of the variable part f , while the right-hand sides of the averaged equations depend only on the variables $\bar{x}^1$ and $\bar{x}^2$. If one substitutes in the averaged equations the expressions for f^k from equations (1) and transforms the relations obtained, taking into account the continuity equation

$$\frac{\partial}{\partial x^1} (\sqrt{g} \bar{B}^1) + \frac{\partial}{\partial x^2} (\sqrt{g} \bar{B}^2) + \frac{\partial}{\partial x^3} (\sqrt{g} \bar{B}^3) = 0 \quad (2)$$

and the equations

$$\overline{\sqrt{g} B^1} = \frac{\partial \bar{A}_3}{\partial x^2}, \quad \overline{\sqrt{g} B^2} = -\frac{\partial \bar{A}_3}{\partial x^1}, \quad (3)$$

where A_3 is the covariant component of the vector potential $\mathbf{A}$, then the equations of the averaged force lines can be represented in the form

$$\frac{d\bar{x}^1}{dx^3} = \frac{1}{\sqrt{\bar{g}} B^3} \frac{\partial \bar{\psi}}{\partial x^2}, \quad \frac{d\bar{x}^2}{dx^3} = -\frac{1}{\sqrt{\bar{g}} B^3} \frac{\partial \bar{\psi}}{\partial x^1}, \quad (4)$$

where

$$\bar{\psi}(\bar{x}^1, \bar{x}^2) = \bar{A}_3 - \frac{1}{\sqrt{\bar{g}} B^3} \sqrt{\bar{g} B^1} \sqrt{\bar{g} B^2}. \quad (5)$$

The integral of these equations, $\bar{\psi}(\bar{x}^1, \bar{x}^2) = \text{const}$, gives the equation of the averaged magnetic surfaces. The true magnetic surfaces, to first order of smallness, are also determined by the equation $\bar{\psi}(\bar{x}^1, \bar{x}^2) = \text{const}$, if in place of the arguments of $\bar{\psi}$ one substitutes $\bar{x}^1 = x^1 - \frac{1}{\sqrt{\bar{g}} B^3} \sqrt{\bar{g} B^1}$, $\bar{x}^2 = x^2 - \frac{1}{\sqrt{\bar{g}} B^3} \sqrt{\bar{g} B^2}$. For the case of magnetic configurations close to symmetric ones, an analogous integral was obtained in (1).

In order to express the integral of the magnetic surfaces directly in terms of the field components B , it is necessary to solve equations (3) with respect to $\bar{A}_3$. The solution of these equations can be written in the form

$$\bar{A}_3 = - \int_{\bar{x}_0^1}^{\bar{x}^1} \sqrt{\bar{g} B^2}(\bar{x}^1, \bar{x}^2) d\bar{x}^1 + \int_{\bar{x}_0^2}^{\bar{x}^2} \sqrt{\bar{g} B^1}(\bar{x}_0^1, \bar{x}^2) d\bar{x}^2, \quad (6)$$

or in an equivalent form differing from (6) by the replacements $x^1 \rightarrow x^2$, $x^2 \rightarrow x^1$, $B^1 \rightarrow -B^2$, $B^2 \rightarrow -B^1$. Integration with respect to one of the arguments in (6) is performed with the second argument fixed, and $\bar{x}_0^1$ and $\bar{x}_0^2$ are arbitrary constants.

If the coordinate x^1 characterizes the deviation from the magnetic axis of a system of nested magnetic surfaces, so that $B^1(0, x^2) = 0$, then, putting $\bar{x}_0^1 = 0$ and hereafter omitting the bars over x^1 and x^2 , we obtain for $\bar{\psi}(x^1, x^2)$ the expression

$$\bar{\psi}(x^1, x^2) = - \int_0^{x^1} \sqrt{g} B^2 dx^1 - \frac{1}{\sqrt{g} B^3} \sqrt{g} \widehat{B^1} \sqrt{g} B^2. \quad (7)$$

The accuracy with which the equation $\psi(x^1, x^2) = \text{const}$ describes the behavior of magnetic surfaces depends essentially on the choice of the variables x^1, x^2, x^3 . For example, in the case of a symmetric field independent of the coordinate x^3 , the equation $\psi = \text{const}$ is the exact equation of the magnetic surfaces. The variables x^i should be chosen in such a way that the necessary effects are "averaged."

2. Stability of magnetic surfaces. The existence of the general solution (7) makes it possible, in particular, to investigate the stability of magnetic surfaces. We shall consider a configuration of magnetic surfaces stable if any small perturbations of the field do not carry a field line far from its initial magnetic surface. Let the unperturbed field B acquire a small increment b ; then, in the linear approximation,

$$\bar{\psi}(x^1, x^2) = - \int_0^{x^1} \sqrt{g} B^2 dx^1 - \frac{1}{\sqrt{g} B^3} \sqrt{g} \widehat{B^1} \sqrt{g} B^2 - \int_0^{x^1} \sqrt{g} b^2 dx^1 - \frac{1}{\sqrt{g} B^3} \left(\sqrt{g} \widehat{B^1} \sqrt{g} b^2 - \sqrt{g} \widehat{B^2} \sqrt{g} b^1 \right). \quad (8)$$

Obviously, magnetic surfaces are unstable if the terms $\bar{\psi}(x^1, x^2)$ that contain no perturbations are identically equal to zero. In this case one can always choose perturbations that will lead to a substantial restructuring of the magnetic surfaces. If, however, the unperturbed part $\bar{\psi}(x^1, x^2)$ vanishes only at individual points x^1, x^2 or on lines $x^1 = x^1(x^2)$, then the presence of a perturbation leads only to small deformations of the magnetic surfaces: to their displacement, the formation of islands and fibers ⁽¹⁾. In this connection it should be taken into account that the form of the function $\bar{\psi}(x^1, x^2)$ depends on the choice of the variables x^i , i.e., on the choice of the lines $x^1 = \text{const}$, $x^2 = \text{const}$, along which the averaging is performed. Since the most dangerous perturbations are those constant along closed lines of force, to reveal the splitting of magnetic surfaces one should average along lines close to closed lines of force.

Fig. 1

Fig. 1

Figure 1: Fig. 1

To obtain estimates of perturbations of magnetic surfaces, let us associate the curvilinear coordinates x^i with the toroidal magnetic surfaces of the unperturbed field $\mathbf{B}$. If the longitudinal magnetic flux $x^1 = \Phi/2\pi$ is taken as the “radial” coordinate, then, since $B^i = \mathbf{B}\nabla x^i$, we have $B^1 = 0$, and the equation of lines of force (1) has the first integral $\Phi = \text{const}$. The second integral can be found from equation (2), in which now only the last two terms are nonzero. Equation (2) is satisfied by introducing the function $\vartheta(\Phi, x^2, x^3)$, defined by the equations

$$\sqrt{g}B^2 = -\partial\vartheta/\partial x^3, \quad \sqrt{g}B^3 = \partial\vartheta/\partial x^2. \quad (9)$$

The second integral of equations (1) is $\vartheta(\Phi, x^2, x^3) = \text{const}$. As the “azimuthal” coordinate x^2 and the “longitudinal” coordinate x^3 , choose coordinates connected with ϑ by the relation $(2, 3)$

$$\vartheta = x^2 - \mu(\Phi)x^3, \quad (10)$$

and require that the line $x^2 = \text{const}$ close after one complete circuit around the torus $\Phi = \text{const}$. According to (9), the unperturbed field in such a coordinate system has components $\sqrt{g}B^i = \{0, \mu, 1\}$. Let the period of variation of x^2 be 2π , and the period of x^3 be L . The quantity $\mu(\Phi)$ is determined by the azimuthal flux $d\chi$ between neighboring magnetic surfaces:

$$d\chi = \frac{1}{2\pi} \int \mathbf{B}\nabla x^2 dV = \frac{1}{2\pi} \int B^2 \sqrt{g} dx^1 dx^2 dx^3 = \frac{\mu d\Phi}{(2\pi)^2} \int dx^2 dx^3, \quad (11)$$

whence $L\mu(\Phi) = 2\pi d\chi/d\Phi$. As is seen from (10), the lines of force $\vartheta = \text{const}$ in the chosen coordinate system are twisted, upon a complete circuit around the torus on each given magnetic surface $\Phi = \text{const}$, through the same angle $L\mu(\Phi)$. In this case the equation of the perturbed magnetic surfaces takes the form

$$\bar{\psi}(x^1, x^2) = - \int_0^\Phi \mu(\Phi) d\Phi - \int_0^\Phi \sqrt{g} b^2 d\Phi = \text{const}. \quad (12)$$

It follows from this that stability with respect to perturbations having a constant component along the line $x^2 = \text{const}$ is determined by the nonvanishing of the quantity μ .

If on some surface $\Phi = \Phi_0$ the mean angle of rotation is $L\mu(\Phi_0) = 2\pi m/n$, where m and n are relatively prime integers, then all field lines of the unperturbed field

on the surface $\Phi = \Phi_0$ close after n circuits of the torus. Let us pass to a new coordinate system $x_*^2 = x^2 - \mu(\Phi_0)x^3$, in which the field lines on the surface $\Phi = \Phi_0$ do not rotate. In the new coordinate system the coordinate lines close after n circuits of the torus and, consequently, $\sqrt{g}B^i$ are periodic functions of x^3 with period nL . The equation of the averaged magnetic surfaces will now be written in the form

$$\bar{\psi}(x^1, x_*^2) = - \int_0^\Phi [\mu(\Phi) - \mu(\Phi_0)] d\Phi - \int_0^\Phi \sqrt{g} b_*^2 d\Phi = \text{const.} \quad (13)$$

According to (13), the stability of magnetic surfaces with respect to field perturbations having a constant component along a closed field line is determined by the nonzero value of the increment $\mu(\Phi) - \mu(\Phi_0)$.

As is seen from (11), the value averaged over the “azimuth” x^2 ,

$$\langle b^2 \sqrt{g} \rangle = \frac{2\pi d\chi_b}{d\Phi L},$$

where $d\chi_b$ is the azimuthal flux of the field $\mathbf{b}$, and therefore, for estimates one may put $L\sqrt{g}b^2 \sim 2\pi d\chi_b/d\Phi$. Taking, moreover, $\Phi \sim B_\parallel \pi \rho^2$, $\chi_b \sim Lb_\perp \rho$, where ρ is the mean radius of the magnetic surface, we obtain the following estimates for the magnitudes of the deformation of magnetic surfaces: $\delta\rho \sim b_\perp/\mu B_\parallel$ in the case (12), and $(\delta\rho)^2 \sim 2b_\perp/\mu'(\rho_0)B_\parallel$ in the case (13). Similar estimates can also be obtained for the case when $\mu'(\Phi_0) = 0$, but $\mu^{(k)}(\Phi_0) \neq 0$. The estimates obtained characterize the order of displacement or splitting of magnetic surfaces.

Since the magnetic axis is also a closed field line, under the influence of perturbations a rosette may form in its neighborhood. To obtain an estimate of the amplitude of its petals, let us take $\mu \sim \mu^{(n)}\rho^n/n!$, $b_\perp \sim b_\perp^{(m)}\rho^m/m!$. Then, if $n+1 > m$, a rosette arises in the neighborhood of the axis, the size of whose petals is of the order

$$\rho_l \sim \left[\frac{n!(n+2)}{m!} \frac{b_\perp^{(m)}}{\mu^{(n)}B_\parallel} \right]^{\frac{1}{n-m+1}}.$$

The typical structure of stable magnetic surfaces in a normal cross section is shown in Fig. 1.

In conclusion we note that the appearance of resonant denominators in calculations of magnetic surfaces and equilibrium plasma configurations (4-6) is due to the occurrence of a fibrous structure. The problem of “small denominators” in application to dynamical systems is considered in work (7).

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