

ON MULTIPLICATIVE REPRESENTATIONS OF BOUNDED ANALYTIC OPERATOR- FUNCTIONS

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Abstract

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MATHEMATICS

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ON MULTIPLICATIVE REPRESENTATIONS OF BOUNDED ANALYTIC OPERATOR-FUNCTIONS

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1. Consider the class $\mathfrak{B}$, consisting of operator-functions $X(\zeta)$, regular for $|\zeta| < 1$, with values in a separable Hilbert space $\mathfrak{H}$, for each of which: 1) $X^*(\zeta)X(\zeta) \leq I$ for $|\zeta| < 1$; 2) $X(0)$ is boundedly invertible; 3) $X(0) - I \in \mathfrak{S}_1$, where $\mathfrak{S}_1$ is the set of operators A with finite trace norm: $|A|_1 = \text{sp}(A^*A)^{1/2} < \infty$ ⁽¹⁾.

Theorem 1. *In order that an operator-function $X(\zeta)$ belong to the class $\mathfrak{B}$, it is necessary and sufficient that the representation*

$$X(\zeta) = \int_0^l \exp\{k[\zeta, \vartheta(t)] dE(t)\} U \Pi(\zeta),$$

$$\Pi(\zeta) = \prod_{j=1}^r \left(\frac{\zeta_j - \zeta}{1 - \bar{\zeta}_j \zeta} \frac{|\zeta_j|}{\zeta_j} P_j + Q_j \right), \quad (1)$$

hold, in which $r \leq \infty$; P_j are orthoprojections, $\dim P_j \mathfrak{H} = m_j < \infty$; $Q_j = I - P_j$; $|\zeta_j| < 1$; $\sum_{j=1}^r m_j(1 - |\zeta_j|) < \infty$; $\vartheta(t)$ is a nondecreasing scalar function ($0 \leq \vartheta(t) \leq 2\pi$); $E(t)$ is a Hermitian increasing operator-function satisfying the condition $\text{sp } E(t) \equiv t$; U is a unitary operator ($I - U \in \mathfrak{S}_1$); $k(\zeta, \vartheta) = (\zeta + e^{i\vartheta})/(\zeta - e^{i\vartheta})$. Moreover, the partial products converge to the Blaschke-Potapov product $\Pi(\zeta)$ in the trace norm, and the integral products also converge to the multiplicative integral.

The last circumstance (which remained unnoted in ⁽⁴⁾, where convergence in the **uniform** norm was considered) makes it possible, in studying the expansion (1), to use the apparatus of determinants of infinite order ⁽¹⁾.

Lemma. *If $X_j(\zeta) \in \mathfrak{B}$ ($j = 1, 2$), then there exist such $Y_j(\zeta) \in \mathfrak{B}$ that*

$$X_1(\zeta)X_2(\zeta) = Y_2(\zeta)Y_1(\zeta), \quad \det X_j(\zeta) = \det Y_j(\zeta).$$

This proposition, formulated for the finite-dimensional case in ⁽⁵⁾, is obtained here by using the possibility of approximating, in the metric $\mathfrak{S}_1$, a function $X(\zeta) \in \mathfrak{B}$ by finite Blaschke products, and also with the aid of the following compactness criterion:

If $A_n, B_n, C_n \in \mathfrak{S}_1$ ($n = 1, 2, \dots$), $A_n \geq 0$, $B_n \geq 0$, and for any $f, g \in \mathfrak{H}$

$$|(C_{nf}, g)|^2 \leq (A_{nf}, f)(B_{ng}, g),$$

then from the compactness in $\mathfrak{S}_1$ of the families $\{A_n\}$ and $\{B_n\}$ there follows the compactness of the family $\{C_n\}$ in $\mathfrak{S}_1$ (cf. ⁽⁴⁾).

* An analogous proposition, established for $\dim \mathfrak{H} < \infty$ by V. P. Potapov ^(2,3) and partially extended to the infinite-dimensional case in ⁽⁴⁾, is also valid for operator-functions of the broader class $\mathfrak{B}$ ($J^* = J$, $J^2 = I$), for which 1) is replaced by the condition $Y^*(\zeta)JY(\zeta) \leq J$.

A nondecreasing function $\vartheta(t)$ ($0 \leq t \leq l$) will be called **canonical** if $0 \leq \vartheta(t) \leq 2\pi$, $\vartheta(t-0) = \vartheta(t)$, and if it assumes the value 2π at no more than one point of the segment $[0, l]$.

Theorem 2. A function $X(\zeta) \in \mathfrak{B}$ admits a representation (1) with a canonical function $\vartheta(t)$, which (as well as the number $l \geq 0$) is uniquely determined by $X(\zeta)$. If t_0 is a point of increase of $\vartheta(t)$, then the operator $E(t_0)$ is also uniquely determined. The Blaschke-Potapov product is determined by $X(\zeta)$ up to a constant left unitary factor.

The first assertion of the theorem follows from Theorem 1, the lemma, and the uniqueness of the multiplicative representation for the function $\det X(\zeta)$; the second follows directly from the general Theorem 3 given below. As for the third assertion, it is proved exactly as was done by V. P. Potapov ⁽³⁾ in the finite-dimensional case. Simple examples show that if t_0 is a point of constancy of $\vartheta(t)$, then $E(t_0)$ is, generally speaking, not determined uniquely by $X(\zeta)$.

Theorem 3. Let

$$X_j(t; \zeta) = \int_0^t \exp\{k[\zeta, \vartheta(\tau)]\} dE_j(\tau) U_j \quad (j = 1, 2; 0 \leq t \leq l, |\zeta| < 1),$$

where $\vartheta(t)$ is a canonical function; U_j are unitary operators; $E_j(t)$ ($0 \leq t \leq l$, $E_j(0) = 0$) are Hermitian operator-functions, continuous and of bounded variation in the uniform operator metric. Then, if

$$X_1(l; \zeta) \equiv X_2(l; \zeta) \quad (|\zeta| < 1),$$

then for every point of increase t_0 of the function $\vartheta(t)$ the equalities

$$X_1(t_0; \zeta) \equiv X_2(t_0; \zeta) \quad (|\zeta| < 1), \quad E_1(t_0) = E_2(t_0)$$

hold.

In the proof, theorems of the Phragmén–Lindelöf type are used, as well as infinite-dimensional analogues of the delicate results of V. P. Potapov ⁽³⁾ on the reconstruction of a multiplicative integral with variable upper limit t from its modulus $R(t)$.

2. Let $X(\zeta) \in \mathfrak{B}$,

$$\det X(\zeta) = e^{ia} \prod_{j=1}^r \frac{\zeta_j - \zeta}{1 - \bar{\zeta}_j \zeta} \frac{|\zeta_j|}{\zeta} \exp \left\{ \int_0^{2\pi} k(\zeta, \vartheta) d\sigma(\vartheta) \right\}$$

$$(\operatorname{Im} a = \operatorname{Im} \sigma(\vartheta) = 0, \quad \vartheta \in [0, 2\pi]).$$

If K is the closed unit disk and C is the unit circle, then we shall call a Borel set $M (\subset K)$ a **carrier** of the function $X(\zeta)$ if $\zeta_j \in M$ ($j = 1, 2, \dots$) and the σ -measure on C is concentrated on $M \cap C$ ($\sigma(M \cap C) = \sigma(C)$).

A function $X_1(\zeta) \in \mathfrak{B}$ is called a (right) divisor of the function $X(\zeta) \in \mathfrak{B}$ if $X(\zeta)X_1^{-1}(\zeta) \in \mathfrak{B}$. In the set of all divisors of the function $X(\zeta)$ one naturally introduces the definitions of greatest common divisor (g.c.d.) and least common multiple (l.c.m.).

Theorem 4. *For every Borel set $M \subset K$, there exists a unique (up to a left unitary factor) divisor $X_M(\zeta)$ of the function $X(\zeta) \in \mathfrak{B}$, possessing the property that M and $K \setminus M$ are carriers of $X_M(\zeta)$ and $X(\zeta)X_M^{-1}(\zeta)$, respectively. Moreover, almost everywhere on the unit circle at least one of the operators $X_M(e^{i\vartheta})$ and $X(e^{i\vartheta})X_M^{-1}(e^{i\vartheta})$ is unitary. If $M_1, M_2, \dots$ are Borel sets in K , then*

$$X_{\cap M_j}(\zeta) = \text{g.c.d.} \{X_{M_j}(\zeta)\}, \quad X_{\cup M_j}(\zeta) = \text{l.c.m.} \{X_{M_j}(\zeta)\}.$$

The first assertion of the theorem in the case when M is an interval follows from the lemma and Theorem 3; then one considers an open set and, finally, an arbitrary Borel set M . After this, the proof of the third assertion presents no difficulty if one uses results, due to Yu. L. Shmul'yan ⁽⁶⁾, on the structure of a semigroup of bounded holomorphic scalar functions. As for the second assertion, in its proof the following fact is essentially used: if $X(\zeta) \in \mathfrak{B}$, then for almost all $\vartheta \in [0, 2\pi]$

$$\lim_{r \rightarrow 1} X^*(re^{i\vartheta})X(re^{i\vartheta}) = X^*(e^{i\vartheta})X(e^{i\vartheta})$$

exists in the trace norm.* This makes it possible to use determinants of infinite order also for the study of boundary properties of functions of the class $\mathfrak{B}$.

The proof of the following proposition, partially formulated for the finite-dimensional case in ⁽⁵⁾, is based on Theorems 1, 3, and 4.

Theorem 5. *An operator-function $X(\zeta)$ belongs to the class $\mathfrak{B}$ if and only if it admits a representation convergent in the metric $\mathfrak{S}$*

$$X(\zeta) = \int_0^{2\pi} \exp\{k(\zeta, \vartheta) d\Sigma_a(\vartheta)\} \cdot \int_0^{2\pi} \exp\{k(\zeta, \vartheta) d\Sigma_s(\vartheta)\} U \Pi_1(\zeta) \Pi(\zeta), \quad (2)$$

$$\Pi_1(\zeta) = \prod_{j=1}^r \int_0^{l_j} \exp\{k(\zeta, \vartheta_j) dE_j(t)\}.$$

*Here $\Pi(\zeta)$ is a Blaschke-Potapov product; $r \leq \infty$; $0 \leq \vartheta_j < 2\pi$; $E_j(t)$ ($0 \leq t \leq l_j$), $\Sigma_s(\vartheta)$, and $\Sigma_a(\vartheta)$ ($0 \leq \vartheta \leq 2\pi$) are Hermitian nondecreasing operator-functions, with $\text{sp } E_j(t) \equiv t$, while $\Sigma_s(\vartheta)$ and $\Sigma_a(\vartheta)$ ($\Sigma_s(0) = \Sigma_a(0) = 0$) are, respectively, singular and absolutely continuous in the metric $\mathfrak{S}_1$; U ($I - U \in \mathfrak{S}_1$) is a unitary operator. In addition, $\Sigma_s(\vartheta)$ is uniquely determined by $X(\zeta)$, and $\Sigma_a(\vartheta)$ by $X^*(e^{i\vartheta})X(e^{i\vartheta})$; the functions $\Pi(\zeta)$ and $\Pi_1(\zeta)$ are determined up to a left unitary factor by $X(\zeta)$. The function $X(\zeta)$ is inner (outer)** if and only if, in the representation (2), $\Sigma_a(\vartheta) \equiv 0$ (respectively, $\Pi(\zeta) \equiv \Pi_1(\zeta) \equiv I$, $\Sigma_s(\vartheta) \equiv 0$), i.e., if and only if $\det X(\zeta)$ is an inner (outer) scalar function.*

Theorems 4 and 5 can be extended to the matrix classes $A^{(n)}$, $D^{(n)}$, $H_\delta^{(n)}$ ($0 < \delta < \infty$), which were the subject of study in ⁽⁵⁾, and also to some of their infinite-dimensional analogues.

3. The theorems presented can be used to study invariant subspaces of certain operators different from normal ones. Below a number of results of this kind are formulated.

Let T be a simple (in other terminology, completely nonunitary) contraction of a Hilbert space $\mathfrak{H}$ ^(8,9a), and suppose $\|T^{-1}\| < \infty$, $I - T^*T \in \mathfrak{S}_1$. It is not hard to see that one can construct the characteristic function $X(\zeta)$ of the operator T ^(8,9b), belonging to the class $\mathfrak{B}$. If M is an arbitrary Borel subset of K , then, as follows from Theorem 4 and the results of B. Sz.-Nagy and C. Foiaş ^(9b, §§3, 4), there exist invariant subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of the operator T such that $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \{0\}$, $\mathfrak{H}_1 \dot{+} \mathfrak{H}_2 = \mathfrak{H}$, and, if T_1 and T_2 are the operators induced by T on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then $X_M(\zeta)$ and $X_{K \setminus M}(\zeta)$ serve as the characteristic functions of the operators T_1 and T_2 , respectively—

* As is known ⁽⁷⁾, a bounded operator function holomorphic in the unit disk has, almost everywhere on the unit circle, in general, only weak boundary values.

** A function $X(\zeta) \in \mathfrak{B}$ is called inner if the operator $X(e^{i\vartheta})$ is unitary for almost all $\vartheta \in [0, 2\pi]$, and outer if $X(\zeta)$ has no nonconstant inner divisors.

respectively (cf. (9^r), Theorem 8). In particular, if M is the interior of the unit disk, then $\mathfrak{H}_1$ is the closed linear span of the finite-dimensional invariant subspaces of the operator T , while the spectrum of the operator T_2 lies on the unit circle (a related result was obtained independently by M. S. Brodskii without using function-theoretic methods).

By choosing the set M in another way, one can arrange, for example, that the function $X_M(\zeta)$ be inner and $X_{K \setminus M}(\zeta)$ outer (see Theorem 5), i.e. that $T_1 \in C_{00}$, $T_2 \in C_{11}$ (9^r).

If some arc of the unit circle is free of the spectrum of the operator T , then, relying on a proposition due to M. S. Brodskii and Yu. L. Shmul'yan ((¹⁰), Theorem 1), one can show that the subspace $\mathfrak{H}_1 = \mathfrak{H}_M$ is determined by the contraction T and by the set M uniquely; moreover, the relations

$$\bigcap \mathfrak{H}_{M_j} = \mathfrak{H}_{\cap M_j}, \quad \overline{\sum \mathfrak{H}_{M_j}} = \mathfrak{H}_{\cup M_j}$$

hold for any collection $\{M_j\}$ of Borel subsets of the disk K .

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