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Abstract

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MATHEMATICS

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ON THE APPROXIMATION OF A PERIODIC FUNCTION BY A BOUNDED SEMIADDITIVE OPERATOR

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1°. In this paper the following problems are considered: 1) the determination of structural properties of a function on the basis of its representation by a bounded semiadditive operator; 2) the determination of the order of approximation of a function by a bounded semiadditive operator, if the structural properties are specified by moduli of smoothness.

Among the numerous works devoted to this question, we mention the papers (1-7).

2°. Notation and assumptions. The function $f(x) \in C_{2\pi}$; $\omega_k(\delta, f)$ is its modulus of smoothness of order k ; $E_n(f)$ is the best approximation by trigonometric polynomials of order $\leq n$; $T_n(x, f)$ is a polynomial of best approximation of order n ; $\sigma(f)$ is the Fourier series:

$$\sigma(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx);$$

$S_n(x, f)$ is the partial sum of the Fourier series of order n ; $\tilde{f}(x)$ is the function trigonometrically conjugate to $f(x)$; $\tau_n(x, f)$ are the Vallée-Poussin sums

$$\tau_n(x, f) = \frac{S_n(x, f) + \dots + S_{2n-1}(x, f)}{n};$$

$T_n(x)$ is a trigonometric polynomial of order $\leq n$; $U(f)$ is a bounded semiadditive (subadditive) operator from $C_{2\pi}$ into $C_{2\pi}$; $\omega_r(\delta)$ satisfies the conditions: 1) it is continuous for $0 \leq \delta < \infty$; 2) $\omega_r(\delta_1) \leq \omega_r(\delta_2)$ when $0 \leq \delta_1 \leq \delta_2$; 3) $\omega_r(0) = 0$; 4) $\omega_r(\lambda\delta) \leq (\lambda + 1)^r \omega_r(\delta)$ for any $\lambda \geq 0$; the function $\Phi_\mu(t)$ ($\Phi_n(t)$): a) is summable and bounded on $[-\pi, \pi]$, b) even, c) positive, d)

$$\int_{-\pi}^{\pi} \Phi_\mu(t) dt = 1;$$

the numbers r , k , and n are natural; $C(0 < C < \infty)$ and $M(0 \leq M < \infty)$ are constants depending only on those arguments which will be indicated,

$$\Delta_{\mu,r} = \int_0^\pi t^r \Phi_\mu(t) dt.$$

3°. Main theorems.

Theorem 1. If $\|U(T_n(x, f))\| \leq M_1 \|T_n^{(k)}(x, f)\|$, then

$$\|U(f)\| \leq (\|U\| + M_1 n^k) E_n(f) + \frac{M_1 n^k}{2^k} \omega_k\left(\frac{\pi}{n}, f\right).$$

Theorem 2. If $M_2 \|T_n^{(k)}(x, f)\| \leq \|U(T_n(x, f))\|$, then

$$M_2 n^k \omega_k(n^{-1}, f) \leq (\|U\| + 2^k M_2 n^k) E_n(f) + \|U(f)\|.$$

Theorem 3. Let $\tilde{f}(x) \in C_{2\pi}$. If

$$\|U(\tau_{[n/2]}(x, f))\| \leq M_3 \|\tau_{[n/2]}^{(k)}(x, \tilde{f})\|,$$

then

$$\|U(f)\| \leq 4\|U\| E_{[n/2]}(f) + 4M_3 n^k E_{[n/2]}(\tilde{f}) + \frac{M_3 n^k}{2^k} \omega_k\left(\frac{\pi}{n}, \tilde{f}\right).$$

Theorem 4. Let $\tilde{f}(x) \in C_{2\pi}$. If

$$M_4 \|T_{[n/2]}^{(k)}(x, \tilde{f})\| \leq \|U(\tau_{[n/2]}(x, f))\|,$$

then

$$M_4 n^k \omega_k(n^{-1}, \tilde{f}) \leq 4\|U\| E_{[n/2]}(f) + 2^{k+2} n^k M_4 E_{[n/2]}(f) + \|U(f)\|.$$

Theorem 5. Let $f^{(r)}(x) \in C_{2\pi}$. If for every $T_n(x)$

$$\|U(T_n)\| \leq M_5 \|\tilde{T}_n^{(k)}(x)\|,$$

then

$$\|U(f^{(r)})\| \leq C_1(r) \|U\| E_n(f^{(r)}) + C_2(r+k) M_5 \sum_{l=0}^{n-1} (l+1)^{r+k-1} E_l(f).$$

Let $U_k(f)$ ($k = 1, 2$) be bounded subadditive operators from $C_{2\pi}$ into $C_{2\pi}$, possessing the following properties for every $T_n(x)$:

- 1) $U_k(T_n)$ is a trigonometric polynomial of order $\leq n$;
- 2) $U_k(T_n) = \widetilde{U}_k(T_n)$;
- 3) for any r , $U_k^{(r)}(T_n) = U_k(T_n^{(r)})$.

Theorem 6. Let p_k ($k = 1, 2$) be natural numbers. If, for every $T_n(x)$:

1)

$$M_6(p_k) \|T_n^{(p_k)}(x)\| \leq \|U_k(T_n)\| \leq M_7(p_k) \|T_n^{(p_k)}(x)\|,$$

then

$$\prod_{k=1}^2 M_6(p_k) \|T_n^{(p_1+p_2)}(x)\| \leq \|U_1(U_2(T_n))\| \leq \prod_{k=1}^2 M_7(p_k) \|T_n^{(p_1+p_2)}(x)\|;$$

2)

$$M_8(p_k) \|\widetilde{T}_n^{(p_k)}(x)\| \leq \|U_k(T_n)\| \leq M_9(p_k) \|\widetilde{T}_n^{(p_k)}(x)\|,$$

then

$$\prod_{k=1}^2 M_8(p_k) \|\widetilde{T}_n^{(p_1+p_2)}(x)\| \leq \|U_1(U_2(T_n))\| \leq \prod_{k=1}^2 M_9(p_k) \|\widetilde{T}_n^{(p_1+p_2)}(x)\|;$$

3)

$$M_{10}(p_1) \|T_n^{(p_1)}(x)\| \leq \|U_1(T_n)\| \leq M_{11}(p_1) \|T_n^{(p_1)}(x)\|,$$

$$M_{10}(p_2) \|\widetilde{T}_n^{(p_2)}(x)\| \leq \|U_2(T_n)\| \leq M_{11}(p_2) \|\widetilde{T}_n^{(p_2)}(x)\|,$$

then

$$\prod_{k=1}^2 M_{10}(p_k) \|\widetilde{T}_n^{(p_1+p_2)}(x)\| \leq \|U_1(U_2(T_n))\| \leq \prod_{k=1}^2 M_{11}(p_k) \|\widetilde{T}_n^{(p_1+p_2)}(x)\|.$$

We note that Theorem 6 generalizes some results of I. P. Natanson ⁽⁸⁻¹⁰⁾.

Theorem 7. Let $0 < C < 1$, r be an even number,

$$I_{\mu}^{(r)}(x, f) = \frac{(-1)^{r/2+1} 2}{C_r^{r/2}} \int_{-\pi}^{\pi} \sum_{k=1}^{r/2} (-1)^{k+r/2} C_r^{k+r/2} f(x+kt) \Phi_{\mu}(t) dt.$$

If

$$1 \leq n \leq \left[\sqrt{\frac{24(1-C)\Delta_{\mu,r}}{r\Delta_{\mu,r+2}}} \right],$$

then

$$Cn^r \Delta_{\mu,r} \omega_r(n^{-1}, f) \leq C_3(r) [(1 + C\Delta_{\mu,r} n^r) E_n(f) + \|I_{\mu}^{(r)}(x, f) - f(x)\|].$$

Theorem 8. Let, for every $T_n(x)$,

$$M_{12} \|\widetilde{T}_n^{(r)}(x)\| \leq \|U(T_n)\|.$$

Then

$$M_{12} n^r \omega_{r+1}(n^{-1}, f) \leq (\|U\| + 2^{r+1} M_{12} n^r) E_n(f) + \|U(f)\|.$$

Theorem 9. If $f^{(k)}(x) \in C_{2\pi}$, then

$$\omega_r(n^{-1}, f^{(k)}) \leq C_4(r, k) [E_n(f^{(k)}) + n^k \omega_{r+k}(n^{-1}, f)].$$

4°. Some applications of the main theorems to singular integrals. We shall apply the main theorems to the V. A. Steklov integral and to some of its modifications.

Theorem 10. Let r be an even number. If

$$\left\| \frac{1}{\delta} \int_0^{\delta} \sum_{l=0}^r (-1)^l C_r^l f \left[x + \left(\frac{r}{2} - l \right) t \right] dt \right\| \leq C_5(r) \omega_r(\delta),$$

then

$$\omega_r(\delta, f) \leq C_6(r) \omega_r(\delta).$$

Theorem 11. Let

$$L_1(h, f) = \left\| \frac{1}{2h} \int_{-h}^h |f(x+t) - f(x)| dt \right\|.$$

Then

$$\omega_1(n^{-1}, f) \leq C_7 \left[n^{-3} L_1(\pi, f) + n^{-3} \int_{n^{-1}}^{\pi} h^{-4} L_1(h, f) dh \right].$$

Corollary. If $L_1(\delta, f) \leq C_8 \omega_1(\delta)$, then $\omega_1(\delta, f) \leq C_9 \omega_1(\delta)$.

We now apply the main theorems to the Jackson integral and some of its generalizations.

Theorem 12. Let r be an even number,

$$D_n^{(r)}(f) = \frac{(-1)^{r/2+1} 2}{C_r^{r/2}} \int_{-\pi}^{\pi} \sum_{k=1}^{r/2} (-1)^{k+r/2} C_r^{k+r/2} f(x+kt) \Phi_n^{(r)}(t) dt,$$

where

$$\Phi_n^{(r)}(t) = \left[\int_{-\pi}^{\pi} \left(\frac{\sin nt/2}{\sin t/2} \right)^{2r+2} dt \right]^{-1} \left(\frac{\sin nt/2}{\sin t/2} \right)^{2r+2}.$$

Then

$$\omega_r(n^{-1}, f) \leq C_{10}(r) \sup_{m \geq n} \|f(x) - D_m^{(r)}(f)\| \leq C_{11}(r) \omega_r(n^{-1}, f).$$

Theorem 13. Suppose that for $0 \leq t \leq n^{-1}$ the kernel $\Phi_n|t| \geq C_{12}n$,

$$\Phi_n(t) = \frac{1}{2\pi} + \sum_{k=1}^n \rho_k(n) \cos kt.$$

Then

$$\omega_2(n^{-1}, f) \leq C_{13} \left\| \int_0^{\pi} |f(x+t) - 2f(x) + f(x-t)| \Phi_n(t) dt \right\|,$$

$$\omega_1(n^{-1}, f) \leq C_{14} \left\| \int_{-\pi}^{\pi} |f(x+t) - f(x)| \Phi_n(t) dt \right\|.$$

Remark. For the Jackson kernel the conditions of the theorem are satisfied.

Theorem 14. Let $\Phi_n(t)$ be a trigonometric polynomial of order $\leq n$,

$$C_{15}n^{-2} \leq \Delta_{n,2} \leq C_{16}n^{-2}, \quad C_{17}n^{-4} \leq \Delta_{n,4} \leq C_{18}n^{-4}.$$

Then

$$\omega_2(n^{-1}, f) \leq C_{19} \sup_{m \geq n} \left\| \int_{-\pi}^{\pi} \{f(x+t) - f(x)\} \Phi_m(t) dt \right\|.$$

All the results given are also valid for the space $L_{2\pi}^p$ ($1 \leq p < \infty$).

5°. Approximation in the space $L_{2\pi}^p$ ($1 < p < \infty$).

We shall adopt the following assumptions. The function* $f(x) \in L_{2\pi}^p$, where $1 < p < \infty$; the numbers $\gamma_k(n)$ are such that

$$|\gamma_{k+1}(n)| \leq C_{20}, \quad \sum_{l=2^k}^{2^{k+1}-1} |\gamma_l(n) - \gamma_{l+1}(n)| \leq C_{21} \quad (k = 0, 1, \dots).$$

Put

$$U_n(x, f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \gamma_k(n) (a_k \cos kx + b_k \sin kx).$$

Theorem 15. Let

$$\rho_k(n) = \frac{[1 - \gamma_k(n)] n^r}{k^m} \quad \text{for } 1 \leq k \leq [n^{r/m}],$$

$$\rho_k(n) = 0 \quad \text{for } k > [n^{r/m}].$$

If

$$|\rho_{k+1}(n)| \leq C_{22}, \quad \sum_{l=2^k}^{2^{k+1}-1} |\rho_{l+1}(n) - \rho_l(n)| \leq C_{23} \quad (k = 0, 1, \dots),$$

then

$$\|U_n(x, f) - f(x)\|_{L^p} \leq C_{24}(p) \omega_m([n^{r/m}]^{-1}, f)_{L^p}.$$

Theorem 16. Let $0 \leq \gamma_{k+1}(n) < 1$ ($k = 0, 1, \dots$),

$$\rho_k(n) = \frac{k^m}{[1 - \gamma_k(n)]n^r} \quad \text{for } 1 \leq k \leq [n^{r/m}],$$

$$\rho_k(n) = 0 \quad \text{for } k > [n^{r/m}].$$

If

$$|\rho_{k+1}(n)| \leq C_{25}, \quad \sum_{l=2^k}^{2^{k+1}-1} |\rho_{l+1}(n) - \rho_l(n)| \leq C_{26} \quad (k = 0, 1, \dots),$$

then

$$\omega_m([n^{r/m}]^{-1}, f)_{L^p} \leq C_{27}(p)[E_{[n^{r/m}]}(f) + \|U_n(x, f) - f(x)\|_{L^p}].$$

As an example of applications of Theorems 15 and 16, we give one theorem.

Theorem 17. Let $0 \leq u < 1$,

$$P_u(x, f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} u^k (a_k \cos kx + b_k \sin kx)$$

be the Poisson operator. Then

$$\omega_1(1 - u, f)_{L^p} \leq C_{28}(p)\|P_u(x, f) - f(x)\|_{L^p} \leq C_{29}(p)\omega_1(1 - u, f)_{L^p}.$$

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* In this section $f(x)$ is not assumed to belong to $C_{2\pi}$.

Note: Figure translations are in progress. See original paper for figures.

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