

SOME PROPERTIES OF ORLICZ SPACES GENERATED BY FUNCTIONS $\varphi(M(x, w))$

V. R. Portnov

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.71213>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.88

MATHEMATICS

V. R. PORTNOV

SOME PROPERTIES OF ORLICZ SPACES GENERATED BY FUNCTIONS $M(x, w)$

(Presented by Academician S. L. Sobolev, January 14, 1966)

Let G be an arbitrary space of points x with a σ -finite measure; let $M(x, w)$ be a function defined on the set $G \times (-\infty, \infty)$, and suppose that for each $w \in (-\infty, \infty)$ it is measurable on G and for almost all (a.a.) $x \in G$ is an N -function on the interval $(-\infty, \infty)$ (see ⁽¹⁾, p. 16). By $M^*(x, w)$ we shall denote the function which, for a.a. $x \in G$, is complementary with respect to $M(x, w)$ (⁽¹⁾, p. 22). This function has the same properties as the function $M(x, w)$, namely: it is measurable on G for each $w \in (-\infty, \infty)$, and for a.a. $x \in G$ is an N -function on the interval $(-\infty, \infty)$.

Definition 1. We shall say that a real-valued function $w(x)$ belongs to the **Orlicz class** $K_M(G)$ if it is measurable on G and

$$g(w; M) = \int_G M(x, w(x)) dx < \infty. \quad (1)$$

Definition 2. We shall say that a sequence of functions $w_n(x)$ from $K_M(G)$ ($n = 1, 2, \dots$) **converges in the mean** to the function $w_0(x) \in K_M(G)$ if

$$\lim_{n \rightarrow \infty} \int_G M(x, w_n(x) - w_0(x)) dx = 0. \quad (2)$$

Definition 3. The **Orlicz space** $L_M(G)$ is the collection of measurable real-valued functions $w(x)$ satisfying the condition

$$\|w\|_M = \sup_{g(v; M^*) \leq 1} \int_G |w(x)| |v(x)| dx < \infty. \quad (3)$$

Definition 4. We shall say that the function $M(x, w)$ **satisfies the $\tilde{\Delta}_2$ -condition** if, for some nonnegative function $\psi_0(x) \in K_M(G)$,

$$\text{vrai sup}_{x \in G} \sup_{w \geq \psi_0(x)} \frac{M(x, 2w)}{M(x, w)} < \infty. \quad (4)$$

Definition 5. We shall say that the function $M(x, w)$ **satisfies the Λ_0 -condition** if there exists a nonnegative function $\delta_0(x) \in L_M(G)$ such that

$$\lim_{\alpha \rightarrow +0} \left(\text{vrai sup}_{x \in G} \left(\sup_{w \geq \delta_0(x)} \frac{M(x, \alpha w)}{\alpha M(x, w)} \right) \right) = 0. \quad (5)$$

We now formulate a number of theorems characterizing properties of Orlicz spaces and classes.

Theorem 1. a) $L_M(G) = K_M(G)$; b) convergence in norm implies convergence in the mean; c) the functional (3) is a norm on $L_M(G)$, and $L_M(G)$ is complete with respect to this norm.

Theorem 2. The inequalities

$$\int_G |w(x)| |v(x)| dx \leq \|w\|_M \|v\|_{M^*} \quad (6)$$

hold for any pair of functions $w(x)$, $v(x)$ from the Orlicz spaces $L_M(G)$, $L_{M^*}(G)$, respectively, and

$$g(w; M) \leq \|w\|_M \quad (7)$$

for any function $w(x)$ from $L_M(G)$ for which $\|w\|_M \leq 1$.

Theorem 3. If the function $M(x, w)$ satisfies the $\widetilde{\Delta}_2$ -condition, then:

a) $L_M(G) = K_M(G)$; b) convergence in the mean is equivalent to convergence in norm; c) the general form of a linear functional $f(w(x))$ on $L_M(G)$ is given by the formula

$$f(w(x)) = \int_G w(x) v(x) dx, \quad (8)$$

where $v(x) \in L_{M^*}(G)$ and is uniquely determined by f .

Theorem 4. If the functions $M(x, w)$ and $M^*(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition, then the space $L_M(G)$ is reflexive.

Theorem 5. If the function $M(x, w)$ satisfies the Δ_0 -condition, then

$$\lim_{\|w\|_M \rightarrow \infty} \frac{g(w; M)}{\|w\|_M} = \infty. \quad (9)$$

Theorem 6. Let the function $M(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition and suppose the following conditions are fulfilled: a) the sequence of functions $\{w_n(x)\}$ ($n = 1, 2, \dots$) converges a.e. on G to the function $w_0(x)$; b) $w_n(x) \in K_M(G)$ for all $n = 1, 2, \dots$; c) for every $\varepsilon > 0$ there exists a measurable set $E_\varepsilon \subset G$ such that $\text{mes } E_\varepsilon < \infty$ and

$$\sup_{n \geq 1} \int_{G \setminus E_\varepsilon} M(x, w_n(x)) dx < \varepsilon;$$

d) for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\sup_{\{E: \text{mes } E < \delta(\varepsilon)\}} \left[\sup_{n \geq 1} \int_E M(x, w_n(x)) dx \right] < \varepsilon.$$

Then $w_0(x) \in K_M(G)$ and

$$\lim_{n \rightarrow \infty} g(w_n(x) - w_0(x); M) = 0.$$

Theorem 7. Let the function $M(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition and

$$\lim_{n \rightarrow \infty} g(w_0(x) - w_n(x); M) = 0,$$

where $w_0(x), w_1(x), w_2(x), \dots$ are functions from $K_M(G)$. Then: a) for every $\varepsilon > 0$ there exists a measurable set $E_\varepsilon \subset G$ such that $\text{mes } E_\varepsilon < \infty$ and

$$\sup_{n \geq 1} \int_{G \setminus E_\varepsilon} M(x, w_n(x)) dx < \varepsilon;$$

b) for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\sup_{\{E: \text{mes } E < \delta(\varepsilon)\}} \left(\sup_{n \geq 1} \int_E M(x, w_n(x)) dx \right) < \varepsilon.$$

We formulate some conditions equivalent to the $\widetilde{\Delta}_2$ -condition.

Theorem 8. In order that the function $M(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition, it is necessary and sufficient that there exist such a nonnegative

a function $\psi_0(x) \in K_M(G)$ such that

$$\text{vrai sup}_{x \in G} \sup_{w \geq \psi_0(x)} \frac{wp(x, w)}{M(x, w)} < \infty,$$

where $p(x, w)$, for a.e. $x \in G$, is the right derivative of the function $M_x(w) = M(x, w)$ with respect to the argument w .

Theorem 9. In order that the function $M^*(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition, it is necessary and sufficient that there exist a measurable nonnegative function $\gamma_0(x)$ on G such that $p(x, \gamma_0(x)) \in K_{M^*}(G)$ ($p(x, w)$ has the same meaning as in Theorem 8) and

$$\text{vrai inf}_{x \in G} \left(\inf_{w \geq \gamma_0(x)} \frac{wp(x, w)}{M(x, w)} \right) > 1.$$

Theorem 10. Suppose: 1) $M(x, w)$ satisfies the $\widetilde{\Delta}_2$ -condition; 2) in G there exists a collection of measurable sets $\{G_n\}$ ($G_n \subset G$, $n = 1, 2, \dots$), no more than countable, such that for any set $E \subset G$ of finite measure and any $\varepsilon > 0$ there is a finite number of sets $G_{n_1}, G_{n_2}, \dots, G_{n_R}$ ($n_1, n_2, \dots, n_R, R$ depend on E and ε) satisfying the inequality

$$\text{mes} \left(\left(E \setminus \bigcup_{k=1}^R G_{n_k} \right) \cup \left(\left(\bigcup_{k=1}^R G_{n_k} \right) \setminus E \right) \right) < \varepsilon.$$

Then the space $L_M(G)$ is separable.

We now give several examples; throughout below the functions $b(x)$, $p(x)$, $q(x)$, $c(x)$, $\beta(x)$, and $\lambda(x)$ are measurable on G , a.e. finite and a.e. positive. Further,

$$\text{vrai inf}_{x \in G} p(x) > 1, \quad \text{vrai sup}_{x \in G} p(x) < \infty.$$

- 1) $M_1(x, w) = b(x)|w|^{p(x)}$ and $M_1^*(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition and the Δ_0 -condition.
- 2) $M_2(x, w) = b(x)M(x)$, where $M(x)$ is an N -function satisfying the Δ_2 -condition for all w (see [1], p. 35). $M_2(x, w)$ satisfies the $\widetilde{\Delta}_2$ -condition. It will also satisfy the Δ_0 -condition if $M^*(w)$ satisfies the Δ_2 -condition for large w (see [6], Theorem 4).
- 3) $M_3(x, w) = b(x)M(w)$, where $M(w)$ is an N -function satisfying the Δ_2 -condition for large w , and moreover $\int_G b(x) dx < \infty$. The function $M_3(x, w)$ satisfies the $\widetilde{\Delta}_2$ -condition. It will also satisfy the Δ_0 -condition if $M^*(w)$ satisfies the Δ_2 -condition for large w (see [6], Theorem 4). Further, the function $M^*(x, w)$ satisfies the Δ_0 -condition; moreover, it will satisfy the $\widetilde{\Delta}_2$ -condition in the case when $M^*(w)$ satisfies the Δ_2 -condition for large w .
- 4) $M_4(x, w) = b(x)|w|^{p(x)} \exp(-c(x)|w|^{-q(x)})$, where

$$2p(x) - 1 \geq q(x),$$

$$\int_G b(x) \exp(-q^{-1}(x))(C(x)q(x))^{p(x)/q(x)} dx < \infty.$$

The functions $M_4(x, w)$ and $M_4^*(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition; moreover, $M(x, w)$ satisfies the Δ_0 -condition.

5) $M_5(x, w) = b(x)|w|^{1+\beta(x)}(1 + |w|^{\beta(x)})^{1/\beta(x)}$, where $\beta(x) \leq 1$ and

$$\int_G b(x)2^{1/\beta(x)} dx < \infty.$$

The functions $M_5(x, w)$ and $M_5^*(x, w)$ satisfy the $\widetilde{\Delta}_2$ -condition.

6) $M_6(x, w) = b(x)|w|\ln^{-1}(1 + \lambda(x)/|w|)$, where

$$\int_G b(x)\lambda(x) dx < \infty.$$

The function $M_6(x, w)$ satisfies the $\widetilde{\Delta}_2$ -condition and the Δ_0 -condition. The function $M_6^*(x, w)$ satisfies the $\widetilde{\Delta}_2$ -condition.

Remark. A. A complete exposition of the theory of Orlicz spaces in the case when $M(x, w) = M(w)$, and G is a closed set of finite measure situated in n -dimensional Euclidean space, is given in ⁽¹⁾. References to the literature on this question are also given there.

B. The case $\text{mes } G = \infty$ and $M(x, w) = M(w)$ is considered in the works ⁽²⁻⁴⁾.

C. Orlicz spaces generated by the functions $M(x, w) = |w|^{p(x)}$, $p(x) > 1$, were studied in ⁽⁵⁾.

D. In the case when $\text{mes } G < \infty$, $M(x, w) = M(w)$, the necessary and sufficient condition for the fulfillment of (9) was obtained for the first time by Ya. B. Rutitskii; the Δ_0 -condition in the case when $M(x, w) = M(w)$, $\delta_0(x) = w_0$, $\text{mes } G < \infty$, is a condition of Ya. B. Rutitskii ⁽⁶⁾.

E. Theorems 6 and 7 for the case of Lebesgue spaces $M(x, w) = |w|^p$ are contained in ⁽⁷⁾.

F. Properties of Orlicz spaces generated by functions $M(x, w)$, different from the properties expressed by Theorems 2-9, have been considered by various authors; here we note only the works ^(8, 9).

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
6 I 1966

REFERENCES

- ¹ M. A. Krasnosel'skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, 1958.
- ² V. I. Sobolev, Tr. Seminar on Functional Analysis, Voronezh State Univ., issue 277 (1956).
- ³ Yu. I. Gribanov, *ibid.*, 6, 29 (1958).
- ⁴ W. Orlicz, Bull. Intern. Acad. Polon. ser. A, Cracovie, (1936).
- ⁵ Z. Birnbaum, W. Orlicz. *Studia math.*, 3, 1 (1931).
- ⁶ Ya. B. Rutitskii, Dokl. AN UkrSSR, No. 10, 1290 (1962).
- ⁷ N. Dunford, J. Schwartz, *Linear Operators*, II, 1962.
- ⁸ H. Nakano, *Modulated Semiorordered Linear Spaces*, Tokio, 1950.
- ⁹ S. Yamamuro, Fac. Sci. Hokkaido Univ., 1, 12, 211 (1953).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.