

A METHOD FOR SOLVING THE GENERAL BOUNDARY-VALUE PROBLEM OF THE PROPAGATION OF LONG AND SUPERLONG RADIO WAVES AROUND THE EARTH

P. E. Krasnushkin

1966

SovietRxiv

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 538.566

MATHEMATICAL PHYSICS

P. E. KRASNYSHKIN

A METHOD FOR SOLVING THE GENERAL BOUNDARY-VALUE PROBLEM OF THE PROPAGATION OF LONG AND SUPER-LONG RADIO WAVES AROUND THE EARTH

(Presented by Academician I. M. Vinogradov on 18 I 1966)

1. The basic features of the waves under consideration are contained in the following boundary-value problem. In an unbounded medium, described by Maxwell's equations in spherical coordinates r, θ, φ with dielectric-constant tensor $\varepsilon((\theta, \varphi, r))$ and divided into 3 regions: 1) the Earth, $0 < r < a(\theta, \varphi)$, where ε degenerates into a scalar $\varepsilon(\theta, \varphi, r) = \varepsilon' + i\varepsilon''$, $\varepsilon'' \neq 0$; 2) the atmosphere, $a(\theta, \varphi) < r < c(\theta, \varphi)$, where $\varepsilon \equiv 1$, and 3) the ionosphere, $c(\theta, \varphi) < r < \infty$, where ε is arbitrary, but as $r \rightarrow \infty$, $\varepsilon \rightarrow \varepsilon = \text{const}$, one seeks the amplitudes $\mathbf{E}(M)$ and $\mathbf{H}(M)$ of the electromagnetic fields produced by a Hertz dipole $P\delta(\theta, r - b)\exp(-i\omega t)$ (δ is the delta function). For simplicity of exposition we assume ε independent of φ and equal to $\varepsilon(\theta, \varphi = \Phi, r)$, where $\varphi = \bar{\varphi} = \text{const}$ is the plane passing through the polar axis and the observation point M . Then, introducing potentials $A(\theta, r)$ and $B(\theta, r)$, related to $H_\varphi = \partial B/r\partial\theta$ and $E_\varphi = \partial A/r\partial\theta$, and their derivatives with respect to θ , $C = \partial B/\partial\theta$ and $D = \partial A/\partial\theta$, we write the boundary-value problem in the form

$$\frac{\partial}{\partial\theta} \cdot \mathbf{B} - L_{r\theta} \cdot \mathbf{B} = P\delta(\theta, r - b), \quad (1)$$

where $\mathbf{B}(\theta, r)$ is a one-column matrix with elements B, A, C, D , through which all components of the amplitudes of the fields $\mathbf{E}$ and $\mathbf{H}$ are expressed, called below a vector-function; $L_{r\theta}$ is a differential operator represented by the 4×4 matrix $\|l_{ji}\|$, where

$$l_{11} = l_{12} = l_{14} = l_{21} = l_{22} = l_{23} = 0; \quad l_{13} = l_{24} = 1;$$

$$l_{31} = -\frac{r^2\Delta}{\varepsilon_{\theta\theta}} \left[\frac{\partial}{\partial r} \left(\frac{\varepsilon_{rr}}{\Delta} \frac{\partial}{\partial r} \cdot \right) + k^2 \cdot \right]; \quad l_{32} = -\frac{ikr^2\Delta}{\varepsilon_{\theta\theta}} \frac{\partial}{\partial r} \left[\frac{\varepsilon^*}{\Delta} \cdot \right];$$

$$l_{33} = -\frac{r^2 \Delta}{\varepsilon_{\theta\theta}} \left[\frac{\partial}{\partial r} \left(\frac{\varepsilon_{\theta r}}{r \Delta} \cdot \right) + \frac{\varepsilon_{r\theta}}{r \Delta} \frac{\partial}{\partial r} \cdot + \operatorname{ctg} \theta \cdot \right]; \quad l_{34} = \frac{i k r \varepsilon^{**}}{\varepsilon_{\theta\theta}}; \quad (2)$$

$$l_{41} = \frac{i k r^2 \bar{\varepsilon}^*}{\Delta} \frac{\partial}{\partial r} \cdot; \quad l_{42} = -r^2 \left[\frac{\partial^2}{\partial r^2} \cdot + k^2 \tilde{\varepsilon} \cdot \right];$$

$$l_{43} = \frac{i k r \bar{\varepsilon}^{**}}{\Delta} \cdot; \quad l_{44} = -\operatorname{ctg} \theta \cdot;$$

$$\Delta = \varepsilon_{\theta\theta} \varepsilon_{rr} - \varepsilon_{\theta r} \varepsilon_{r\theta}; \quad \tilde{\varepsilon} = \varepsilon_{\varphi\varphi} + (\varepsilon_{\varphi\theta} \varepsilon^* + \varepsilon_{\varphi r} \varepsilon^{**}) / \Delta;$$

$$\varepsilon^* = \varepsilon_{r\varphi} \varepsilon_{\theta r} - \varepsilon_{\theta\varphi} \varepsilon_{rr}; \quad \varepsilon^{**} = \varepsilon_{r\theta} \varepsilon_{\theta\varphi} - \varepsilon_{\theta\theta} \varepsilon_{r\varphi}; \quad (3)$$

$$\bar{\varepsilon}^* = -\varepsilon_{\varphi r} \varepsilon_{r\theta} + \varepsilon_{\varphi\theta} \varepsilon_{rr}; \quad \bar{\varepsilon}^{**} = \varepsilon_{\theta r} \varepsilon_{\varphi\theta} - \varepsilon_{\theta\theta} \varepsilon_{\varphi r},$$

under the conditions of continuity of $\mathbf{B}$ on the interfaces of the regions $r = a$ and $r = c$, and boundedness as $r \rightarrow 0$, $r \rightarrow \infty$, $\theta \rightarrow 0$, $\theta \rightarrow \pi$.

Equation (1) is approximate. In it an integral has been omitted, under whose sign stand products of the elements of $\mathbf{B}$ by derivatives of the components of ε with respect to θ .

§ 2. Method of coupled lines. We seek the solution of (1) in the form

$$\mathbf{B} = \sum \mathbf{Y}_k(r) \cdot \mathbf{b}_k(\theta), \quad (4)$$

where $\mathbf{b}_k$ is a vector function with elements b_k, a_k, c_k, d_k , and $\mathbf{Y}_k$ is a vector function with elements Y_k, Z_k, U_k, V_k , which is an eigenfunction of the operator $L_{r\bar{\theta}}$, obtained from $L_{r\theta}$ by “freezing” the coefficients in $\|v_{ji}\|$ with respect to θ ; θ enters into $L_{r\bar{\theta}}$ as a parameter. The $\mathbf{Y}_k$ are determined from the equation

$$L_{r\bar{\theta}} \cdot \mathbf{Y} = \lambda \mathbf{Y} \quad \text{under the conditions} \quad (\mathbf{Y}_k, \mathbf{Y}_r^*) = \delta_{kr}; \quad (5)$$

$\lambda(\theta)$ are the eigenvalues of $L_{r\bar{\theta}}$; they lie in the 2nd and 4th quadrants of the λ -plane. We number them in order of increasing modulus, respectively $k = 1, 2, \dots$ and $k = -1, -2, \dots$. Parentheses in (5) denote scalar products, and the asterisk denotes eigenfunctions of the adjoint operator. Introduce $L_{r\bar{\theta}}$ into (1):

$$\frac{\partial}{\partial \theta} \cdot \mathbf{B} - L_{r\bar{\theta}} \cdot \mathbf{B} + [L_{r\bar{\theta}} - L_{r\theta}] \cdot \mathbf{B} = \mathbf{P} \delta(\theta, r - b). \quad (6)$$

Substituting (4) into (6), multiplying scalarly by $\mathbf{Y}_r^*$, and taking (5) into account, we obtain a system of coupled-line equations (1), integrable on a computer:

$$\frac{d}{d\theta} \cdot \mathbf{b}_k - i\nu_k(\theta)\mathbf{b}_k + \sum_{r=-\infty}^{\infty} \|S_{ji}^{(k,r)}(\theta)\| \cdot \mathbf{b}_r = P_k \delta(\theta); \quad \lambda_k = i\nu_k. \quad (7)$$

S_{ji} are 4×4 matrices of complex numbers formed from the scalar products of $[L_{r\bar{\theta}} - L_{r\theta}] \cdot \mathbf{Y}_k$ with $\mathbf{Y}_r^*$. If the spectrum of $L_{r\bar{\theta}}$ has a continuous part, then an integral over ν also enters into (7). For small S_{ji} and $\nu_k \neq \nu_r$, i.e., far from the cones of spatial resonance⁽¹²⁾, neglecting S_{ji} , we obtain the solution in the form of a sum of modulated normal waves*

$$\mathbf{B} \exp(-i\omega t) = \sum_{k=-\infty}^{+\infty} C_k \mathbf{Y}_k(r, \theta) \exp \left[-i \left(\omega t - \int \nu_k d\theta \right) \right]. \quad (4')$$

The S_{ji} in (7) take into account the interaction of normal waves on sections of the path where ε depends on θ , for example in the sunrise and sunset belt; and also because of the inhomogeneity of the metric of space in spherical coordinates. It should be noted that, in a nonvertical magnetic field of the Earth $\mathbf{H}_0$, when $\varepsilon_{\theta r}$ and $\varepsilon_{r\varphi}$ are not equal to 0, $\nu_k \neq -\nu_{-k}$, and therefore the reciprocity principle in wave propagation with respect to θ is violated (see⁽¹¹⁾).

§ 3. To find the wave numbers ν_k , it is convenient to eliminate C and D from (1) and pass to the coupled-line equations containing second-order derivatives with respect to θ . Then (5) becomes the following boundary-value problem for eigenvalues of the parameter ν :

$$\begin{aligned} Y''_{rr} + aY'_r + bY + cZ'_r + dZ &= 0; \\ Z''_{rr} + eZ + fY'_r + gY &= 0, \end{aligned} \quad (5')$$

where a prime denotes differentiation with respect to r ;

$$\begin{aligned} a &= \frac{\Delta}{\varepsilon_{rr}} \left[\left(\frac{\varepsilon_{rr}}{\Delta} \right)'_r \pm ik \frac{(\varepsilon_{\theta r} + \varepsilon_{r\theta})}{\Delta} S \right] \\ b &= \frac{k^2 \Delta}{\varepsilon_{rr}} \left[1 - \frac{\varepsilon_{\theta\theta}}{\Delta} S^2 \pm \frac{ir}{k} \left(\frac{\varepsilon_{\theta\theta}}{r\Delta} \right)'_r S \right]; \\ c &= \frac{ik\varepsilon^*}{\varepsilon_{rr}}; \quad d = ik \left(\frac{\varepsilon^*}{\Delta} \right)'_r \frac{\Delta}{\varepsilon_{rr}} \pm \frac{k^2 \varepsilon^{**}}{\varepsilon_{rr}} S; \quad e = k^2(\tilde{\varepsilon} - S^2); \end{aligned}$$

* Normal waves ^(1-4,11) are also called free, proper, and, in English, residue waves and modes ⁽⁵⁻⁷⁾, which is not entirely apt and is sometimes translated as “modes.”

$$f = -\frac{ik\varepsilon^*}{\Delta}; \quad g = \pm \frac{\bar{k}^2 \varepsilon^{**}}{\Delta} S; \quad S = \frac{\nu}{kr}.$$

It is necessary to find ν that ensures a nonzero solution of (5') under the boundedness conditions $|Y|$ and $|Z|$ as $r \rightarrow 0$ and $r \rightarrow \infty$. On an electronic computer it is convenient to solve (5') by the method of sweeping the conditions at $r \rightarrow 0$ and $r \rightarrow \infty$ to some point $r = \bar{r}$, where Y, Z, Y_r', Z_r' must be continuous. Such a method is equivalent to the more economical method of sweeping and matching the impedances $|z|$ of the surface $\bar{r} = \text{const}$ for the regions $(\bar{r}, \infty)$ and $(\bar{r}, 0)$, denoted by $|z(\bar{r} + 0)|$ and $|z(\bar{r} - 0)|$, and determined by the expressions

$$E_\theta = z_{11}'(\bar{r} \pm 0)H_\theta + z_{12}'(\bar{r} \pm 0)H_\varphi; \quad E_\varphi = z_{21}'(\bar{r} \pm 0)H_\theta + z_{22}'(\bar{r} \pm 0)H_\varphi;$$

$|z|$ are expressed in terms of Y, Y_r', Z , and Z_r' . The matching equation for $|z|$ will be:

$$\det \begin{vmatrix} z_{11}'(\bar{r} + 0) - z_{11}'(\bar{r} - 0) & z_{12}'(\bar{r} + 0) - z_{12}'(\bar{r} - 0) \\ z_{21}'(\bar{r} + 0) - z_{21}'(\bar{r} - 0) & z_{22}'(\bar{r} + 0) - z_{22}'(\bar{r} - 0) \end{vmatrix} = 0. \quad (8)$$

Its roots will be ν_k . We choose as $\bar{r}$ the surface $\bar{r} = c$. To find $|z|$ from (5'), introduce the impedance functions u, χ : $Y = \exp \int u dr$, $Z = \chi Y$; substituting them into (5'), we obtain

$$u_r' + u^2 + au + b + c(\chi_r' + u\chi) + d\chi = 0,$$

$$\chi_{rr}'' + 2u\chi_r' + u^2\chi + u_r'\chi + e\chi + fu + g = 0. \quad (9)$$

To compute $|z^\nu(c + 0)|$, we integrate (9) on the computer from r_∞ to $r = c$, where $r_\infty \gg c$ is chosen in the region in which ε is practically constant. The initial values $u(r_\infty)$ are obtained from equation (9) for $u_r' = \chi_r' = \chi_{rr}'' = 0$:

$$u^4 + \bar{a}u^3 + (\bar{e} + \bar{b} - \bar{c}\bar{f})u^2 + (\bar{e}\bar{a} - \bar{c}\bar{g} - \bar{d}\bar{f})u + (\bar{e}\bar{b} - \bar{d}\bar{g}) = 0, \quad (10)$$

which passes into Booker's equation [8] as $r \rightarrow \infty$, if the relation of ε to the plasma parameters is determined from the Lorentz equation (formula (1) from [11]). Of the four roots of (10) we choose $u^o(r_\infty)$ and $u^e(r_\infty)$, corresponding to ordinary and extraordinary waves decaying at $+\infty$. The initial $\chi(r_\infty)$ for these u are found from the formula $\chi = -(\bar{g} + \bar{f}u)/(\bar{e} + u^2)$. Integration of (9) from

$r = r_\infty$ in the direction of decreasing r , i.e., toward the traveling waves, signifies a continuous transformation of the impedances of waves of types o and e

$$Z_y^{e,o}(r) = \frac{E_\theta}{H_\varphi} = \frac{1}{\Delta} \left[\frac{\varepsilon_{rr}}{ik} u^{e,o}(r) + \varepsilon^* \chi^{e,o}(r) \pm \varepsilon_{\theta r} S \right];$$

$$Z_z^{e,o}(r) = \frac{H_\theta}{E_\varphi} = -\frac{1}{ik} \left[u^{e,o}(r) + \frac{\chi_{r'}}{\chi^{e,o}} \right]; \quad X^{e,o}(r) = \frac{E_\varphi}{H_\varphi} = \chi^{e,o}(r) \quad (11)$$

from the adiabatic values at $r = r_\infty$ to the values on the surface $r = \bar{r}$. These impedances are related to the previously introduced impedances z_{jk} of the surface $r = \bar{r}$ by the expressions

$$z_{11} = (Z_y^e - Z_y^o)/\delta; \quad z_{12} = (X^e Z_z^e Z_y^o - X^o Z_z^o Z_y^e)/\delta;$$

$$z_{21} = (X^e - X^o)/\delta, \quad z_{22} = (X^e - X^o) Z_z^e Z_z^o/\delta, \quad (12)$$

where $\delta = X^e Z_z^e - X^o Z_z^o$. Carrying the integration to the point $r = c$, we obtain, from formulas (11), (12), the impedances $z_{jk}(c+0)$. The impedances $z_{jk}(c-0)$ are found in two stages. First, for two values $u^{o,e}(r_0)$, $\chi^{o,e}(r_0)$, where $r_0 \ll a$, corresponding to waves traveling toward the center of the Earth, one integrates (9) from $r = r_0$ to the Earth's surface $r = a$ and finds, analogously to the case of the ionosphere, $z_{12}(a-0) = Z_y^e = Z_y^o$ and $z_{21}^{-1}(a-0) = Z_z^e = Z_z^o$. Because of the isotropy of the Earth, $z_{11}(a-0) = z_{22}(a-0) = 0$. The transformation of impedances over the next interval (a, c) is carried out analytically using the formulas

$$iz_{12}^\nu(c-0) = \frac{D_\nu(a', c') - iz_{12}^\nu(a-0)D_\nu(a, c')}{D_\nu(a', c) - iz_{12}^\nu(a-0)D_\nu(a, c)} = \frac{\overline{D_\nu(a', c')}}{\overline{D_\nu(a', c)}}, \quad (13)$$

$$iz_{21}^\nu(c-0) = \frac{D_\nu(a, c) - iz_{21}^\nu(a-0)D_\nu(a', c)}{D_\nu(a, c') - iz_{21}^\nu(a-0)D_\nu(a', c')} = \frac{\overline{D_\nu(a, c)}}{\overline{D_\nu(a, c')}},$$

where $D_\nu(a, c)$ are two-argument functions formed from products of the modified Hankel functions $h_\nu^{(1,2)}(ka)$ and $h_\nu^{(1,2)}(kc)$ ⁽²⁻⁴⁾; $z_{11}^\nu(c-0)$ and $z_{22}^\nu(c-0)$ are equal to zero. Substituting (11), (12), (13) into (8), we obtain the final equation for determining the wave numbers of the normal waves

$$[\overline{D_\nu(a', c')} - iZ_y^e \overline{D_\nu(a', c)}][D_\nu(a, c') + iZ_z^o D_\nu(a, c)] -$$

$$-(X^e/X^o)[D_\nu(a', c') - iZ_y^o D_\nu(a', c)] [\overline{D_\nu(a, c')} + iZ_z^e \overline{D_\nu(a, c)}] = 0. \quad (14)$$

Equation (14), together with (9) and (11), was used in ^(2,3) for the case of vertical $\mathbf{H}_0$, when (10) becomes biquadratic. The values of $\bar{u}$ and χ for this case are given by formulas (2.40)–(2.41) in ⁽³⁾. If the ionosphere is homogeneous, i.e., $N_e = 0$ for $r < c$ and $N_e = \bar{N}_e$ for $r > c$, then there is no need to integrate (9), since the impedances are obtained directly from (11):

$$Z_y^{e,o}(c) = \frac{1}{\Delta} \left[\frac{\varepsilon_{rr}}{ik} \bar{u}^{e,o} + \varepsilon^* \bar{\chi}^{e,o} \mp \varepsilon_{\theta r} S \right]; \quad Z_z^{e,o} = -\frac{1}{ik} \bar{u}^{e,o}; \quad X^{e,o} = \bar{\chi}^{e,o}. \quad (15)$$

Equation (14) with these impedances was used in work ⁽⁴⁾ for the case of the vertical magnetic field of the Earth, when $\varepsilon_{\theta r} = 0$, $\varepsilon^* = -\varepsilon_{\theta\varphi} \varepsilon_{rr}$, and $\Delta = \varepsilon_{\theta\theta} \varepsilon_{rr}$. The values of $\bar{u}^{e,o}$ and $\bar{\chi}^{e,o}$ in this case are determined from the biquadratic equation (10) by formulas (2.40)–(2.41) of work ⁽³⁾. In works ⁽²⁻⁴⁾, in a neighborhood of $\nu = ka$ and $\nu = kc$, the functions $h_\nu^{(1)}$ and $h_\nu^{(2)}$ in (14) were approximated by Airy functions:

$$h_\nu^{(1,2)}(z) = z^{1/6} h_{1,2}(\zeta); \quad \zeta = -\eta(\nu - z); \quad \eta = \sqrt[3]{z/2} \quad (16)$$

by the “comparison-equation” method ⁽⁹⁾. In the first of the works ⁽⁴⁾, where hand calculation was used, we used tables of the functions $h_{1,2}(\zeta)$ ⁽¹⁰⁾ with 8 significant figures for the complex argument ζ in the circle $|\zeta| \leq 6$. In ^(2,3), for calculating $h_{1,2}(\zeta)$, series given in the foreword to ⁽¹⁰⁾ were used.

Introducing ν_k and Y_k into (4'), we obtain the first approximation to the solution of the boundary-value problem. The solution (7), with allowance for the terms S_{ji} , will give the second approximation. To refine the interaction of the normal waves, one must take into account the integral terms omitted in (7).

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
27 XII 1965

CITED LITERATURE

- ¹ P. E. Krasnushkin, Doctoral dissertation, Moscow State University, 1945.
- ² P. E. Krasnushkin, DAN, **138**, 1055 (1961).
- ³ P. E. Krasnushkin, *Nuovo Cim.*, **26**, X, 50 (1962).
- ⁴ P. E. Krasnushkin, N. A. Yablochkin, *Theory of the Propagation of Superlong Radio Waves*, etc., Moscow, 1955; 2nd ed., Computing Center of the Academy

of Sciences of the USSR, 1963.

⁵ I. R. Wait, *Canad. J. Res.*, **41**, 299 (1963).

⁶ I. R. Wait, K. Spies, *Geophys. Res.*, **65**, 8, 2325 (1960).

⁷ K. G. Budden, *Proc. Roy. Soc.*, A **227**, 516 (1955); Collection of Papers on Geophysics, 1964, p. 56.

⁸ H. G. Booker, *Phil. Trans. Roy. Soc.*, A **237**, 411 (1939).

⁹ P. E. Krasnushkin, *Vestn. Mosk. Univ.*, **6**, 73 (1948).

¹⁰ W. H. Furry, H. H. Aiken, *Tables of Modified Hankels Functions of Order One Third*, USA, Cambridge, 1945.

¹¹ P. E. Krasnushkin, R. B. Baibulatov, *DAN*, **171**, No. 2 (1966).

¹² P. E. Krasnushkin, *Physical Encyclopedic Dictionary*, **3**, Normal Waves, 1963.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.