

ON LINEAR EQUATIONS IN SPACES OF TEST AND GENERALIZED FUNCTIONS

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Abstract

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MATHEMATICS

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ON LINEAR EQUATIONS IN SPACES OF TEST AND GENERALIZED FUNCTIONS

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1°. Let Ψ_1, Ψ_2 be countably normed (c.-n.) spaces*; Ψ'_1, Ψ'_2 their conjugates; A a linear continuous** operator mapping Ψ_1 into Ψ_2 . We shall use the following notation: $\mathfrak{R}(A)$ is the set of values of the operator A ; $N(A)$ is the subspace of solutions of the equation $Ax = 0$; $\alpha(A) = \dim N(A)$; $\beta(A) = \dim \Psi_2/\mathfrak{R}(A)$.

The operator A is called **normally solvable** if the equation $Ax = y$ is solvable if and only if $\varphi(y) = 0$ for all $\varphi \in N(A^*)$, where A^* is the operator conjugate to A . The property of normal solvability of the operator A is equivalent ⁽²⁾ to the closedness of the set $\mathfrak{R}(A)$. Following ⁽³⁾, a normally solvable operator A will be called a Φ_+ -(Φ_-)-operator if the number $\alpha(A)$ ($\beta(A)$) is finite, and a Φ -operator if both numbers $\alpha(A), \beta(A)$ are finite.

Theorem 1. *In order that A be a Φ_+ -operator, it is necessary and sufficient that there exist a continuous operator B mapping Ψ_2 into some c.-n. space Ψ_3 and such that BA is a Φ_+ -operator.*

Proof. The necessity is obvious. One may put $B = I$, where I is the identity operator of the space Ψ_2 .

Sufficiency. Denote $BA = S$. Clearly, $\alpha(A) \leq \alpha(S) < \infty$. Let

$$\|\varphi\|_1^{(i)} \leq \dots \leq \|\varphi\|_p^{(i)} \leq \dots$$

be a countable system of norms of the space Ψ_i ($i = 1, 2, 3$). Decompose Ψ_1 into the direct (topological) sum

$$\Psi_1 = N(A) \dot{+} N_1 \dot{+} \Psi_0,$$

where N_1 is a direct complement of the subspace $N(A)$ to $N(S)$. Define the operator A_1 as the restriction of the operator A to the subspace*** $\Psi_0 \dot{+} N_1$, and prove that the operator A_1^{-1} , defined on $\mathfrak{R}(A)$, is bounded.

Let

$$\mathfrak{R}(A) \supset F = \{f \in \mathfrak{R}(A) : \|f\|_p^{(2)} \leq C_p \ (p = 1, 2, \dots)\}$$

be a bounded set. For any element $f \in F$ we have $Bf = SA_1^{-1}f$, $A_1^{-1}f = \xi_f + \eta_f$, where $\xi_f \in \Psi_0$, $\eta_f \in N_1$. The restriction S_1 of the operator S to Ψ_0 maps one-to-one the c.-n. space Ψ_0 onto the c.-n. space $\Re(S)$. Hence (see ⁽¹⁾, § 7), the operator S_1^{-1} is bounded, and the set

$$\{\xi_f = S_1^{-1}Bf, f \in F\}$$

is bounded in Ψ_1 .

Let $r = \dim N_1$ (we may assume that $r \geq 1$) and let $x_1, x_2, \dots, x_r$ be a basis of the subspace N_1 . For any $f \in F$ we have

$$\eta_f = \sum_{i=1}^r a_i^f x_i,$$

where a_i^f are constants depending on $f \in F$. We shall prove that $|a_i^f| \leq C = \text{const}$ ($i = 1, \dots, r$), where C does not depend on f . Fix an arbitrary number $p_0 \geq 1$. The elements $y_i = A_1 x_i \in \Psi_2 \subset \Psi_{2p_0}$ (see ⁽¹⁾) are linearly independent,

* By a countably normed space we shall mean a complete countably normed space.

** In a c.-n. space the notions of continuity and boundedness of an operator coincide ⁽¹⁾.

*** By a subspace we shall mean a closed linear submanifold.

and therefore (4) for some $M > 0$

$$\sum_{i=1}^r |a_i^f| \leq M \left\| \sum_{i=1}^r a_i^f y_i \right\|_{p_0}^{(r)} = M \|f - A_1 \xi_f\|_{p_0}^{(r)} \leq M(C_{p_0} + C'_{p_0}) = C.$$

Hence we obtain $\|\eta_f\|_p^{(1)} \leq C_p''$ for any p , i.e., the set $A_1^{-1}F \subset \Psi_1$ is bounded. Thus the operator A_1^{-1} is bounded and, consequently, continuous. It follows that the set $\Re(A_1) = \Re(A)$ is closed.

Remark 1. In the particular case when Ψ_1, Ψ_2 are Banach spaces, $\Psi_3 = \Psi_1$, and $BA = I + T$ (T is a completely continuous operator), Theorem 1 was proved earlier by S. G. Mikhlin ⁽⁵⁾.

From Theorem 1 one easily obtains the following result.

Theorem 2. *In order that A be a Φ -operator, it is necessary and sufficient that there exist a continuous operator B such that $\alpha(B)$ is finite and BA is a Φ -operator.*

Remark 2. Another characterization of the set of Φ -operators is given in Theorem 12 of the paper ⁽²⁾. From the proof of necessity in that theorem it

follows that if A is a Φ -operator from Ψ_1 into Ψ_2 , then there exists a continuous operator B from Ψ_2 into Ψ_1 such that $\alpha(B)$ is finite and $BA = I - L$, where L is a finite-dimensional operator in Ψ_1 . The converse assertion is true by virtue of Theorem 2.

2°. Let now Ψ be some basic space (in the sense of ⁽¹⁾, Ch. II, § 1) such that it is a c.n. space and every function $\varphi(x) \in \Psi$ determines a certain regular functional on Ψ , in such a way that different functions correspond to different functionals. Examples of such spaces are the space C^∞ , considered in 3°, and also any space of type $K\{M_p\}$ ⁽¹⁾ possessing the property: for some p the function $M_p^{-2}(x)$ is summable.

Let A be a linear continuous operator acting in the space Ψ and such that $A^*\varphi \in \Psi$ for any function $\varphi \in \Psi$. Elements $\varphi \in \Psi$ are called **basic functions**, and elements $f \in \Psi'$ are **generalized functions** (g.f.) (over the space Ψ). Let $f \in \Psi'$. We shall say that $u \in \Psi'$ is a **generalized solution** of the equation

$$Au = f, \quad (1)$$

if $(u, A^*\psi) = (f, \psi)$ for any function $\psi \in \Psi$. If $f \in \Psi$ and $u \in \Psi$ satisfies equation (1), then the function u will be called a **classical solution** of equation (1). For convenience we shall denote by A_0^* the operator A^* considered on the basic space Ψ .

Definition. The operator A will be called **normally solvable** in g.f. if equation (1) is solvable (in g.f.) if and only if $(f, \varphi) = 0$ for all $\varphi \in N(A_0^*)$.

Theorem 3. If A_0^* is a Φ -operator, then the operator A is normally solvable in g.f.

Proof. Let $\{\varphi_j\}_{j=1}^s$ be a basis of the subspace $N(A_0^*)$; let $\{y_k\}_{k=1}^q$ be a complete system of generalized solutions of the homogeneous equation (1) (the numbers s, q are finite by assumption). Introduce into consideration the subspace $\Psi_0 = \mathfrak{R}(A_0^*)$. By the condition of Theorem 3, $\varphi \in \Psi_0$ if and only if $(y_k, \varphi) = 0$ ($k = 1, \dots, q$). Construct elements $\xi_i \in \Psi$ ($i = 1, \dots, q$), biorthogonal to the g.f. y_k , and let Ψ_1 be the subspace with basis $\xi_1, \dots, \xi_q$. Then for the space Ψ we have the representation $\Psi = \Psi_0 + \Psi_1$. Denote by P_0 the (continuous) projector of the space Ψ onto Ψ_0 .

Suppose the conditions are satisfied: $(f, \varphi_j) = 0$ ($j = 1, \dots, s$). Construct the functional u_0 , putting* for each $\varphi \in \Psi$

* We apply the well-known device (see ⁽⁶⁾) for constructing a particular solution of an equation of the form (1).

$$(u_0, \varphi) = (f, \varphi^{(0)}). \quad (2)$$

Here $\varphi^{(0)} \in \Psi$ is a solution of the equation $A_0^* \psi = P_0 \varphi$. Let us prove that u_0 is a uniquely determined linear continuous functional on Ψ . Indeed, let A_1^* be the restriction of the operator A_0^* to the direct complement of the subspace $N(A_0^*)$ in the space Ψ . The operator $B = (A_1^*)^{-1}$ is continuous and

$$\varphi^{(0)} = BP_0 \varphi + \sum_{j=1}^s a_j \varphi_j.$$

Consequently, $(u_0, \varphi) = (f, BP_0 \varphi)$. It is easy to see that u_0 is a solution of equation (1).

3°. We now apply the results from 1° and 2° to the investigation of a system of singular integral equations with Cauchy kernel

$$\mathfrak{A}\varphi \equiv A(t)\varphi(t) + B(t)S\varphi + \int_{\Gamma} K(t, \tau)\varphi(\tau) d\tau = f(t) \quad (3)$$

(where by S we shall denote the singular operator

$$\frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau) d\tau}{\tau - t},$$

the symbolic matrices $C(t) = A(t) + B(t)$, $D(t) = A(t) - B(t)$ of which degenerate at a finite number of points of the contour Γ . As the basic space we take the space C^∞ of functions given and infinitely differentiable on Γ .* We shall assume that $A(t), B(t) \in C^\infty$, $K(t, \tau)$ is infinitely differentiable on $\Gamma \times \Gamma$, and Γ is a closed smooth, infinitely differentiable contour bounding a finite simply connected domain.

It is known⁽⁷⁻⁹⁾ that in the spaces H (see⁽⁷⁾) and $L_2(\Gamma)$ the Noether theorems are valid if and only if $\Delta_1(t) \equiv \det D(t) \neq 0$, $\Delta_2(t) \equiv \det C(t) \neq 0$ on Γ . Nevertheless, the following is valid.

Theorem 4. *If each of the functions $\det(A - B)$, $\det(A + B)$ has no more than a finite number of zeros of integral multiplicity on the contour, then for equation (3) the Noether theorems are valid in the space C^∞ .*

Proof. It is clear that the operator $\mathfrak{A}$ maps C^∞ continuously into itself. For simplicity of notation we shall assume that $\Delta_l(t)$ ($l = 1, 2$) has one root $t = \alpha_l$ of multiplicity m_l . Consider the operator $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, where**

$$\mathfrak{B}_l \equiv \frac{I - P_l}{2\Delta_l} H_l [I + (-1)^l S], \quad P_l \psi = \sum_{\nu=0}^{m_l-1} \frac{1}{\nu!} \psi^{(\nu)}(\alpha_l) (t - \alpha_l)^\nu.$$

Here H_1, H_2 are matrices adjugate to D, C , respectively; the operators $\mathfrak{B}_1, \mathfrak{B}_2$ are closed and defined on all of C^∞ , hence⁽¹⁰⁾ they are continuous. It is easy to see that

$$\mathfrak{B}\mathfrak{A}\varphi = \varphi(t) + \int_{\Gamma} T(t, \tau)\varphi(\tau) d\tau,$$

where $T(t, \tau)$ is a matrix of the type $K(t, \tau)$. $\mathfrak{B}\mathfrak{A}$ is a Φ -operator (with index equal to zero) in the space $\widetilde{H}$ (see (7)) and, consequently, in the space C^∞ , since every solution of the equation $\mathfrak{B}\mathfrak{A}\varphi = \mathfrak{B}f$ ($f \in C^\infty$) from the class $\widetilde{H}$ belongs to C^∞ and $N((\mathfrak{B}\mathfrak{A})^*) \subset C^\infty$. The conditions of Theorem 2 are fulfilled, and therefore $\mathfrak{A}$ is a Φ -operator in the space C^∞ .

From what was said above, on the basis of Theorem 2 and remark (11), we obtain the following.

Corollary. *If the conditions of Theorem 4 are fulfilled, then the operator $\mathfrak{A}$ admits a bounded regularization in the space C^∞ .*

Theorem 5. *If the conditions of Theorem 4 are fulfilled, then the operator $\mathfrak{A}$ is normally solvable in o.f. over the space C^∞ .*

* We introduce the topology in C^∞ in the usual way:

$$\|\varphi\|_p = \max_{\substack{0 \leq k \leq p \\ t \in \Gamma}} |\varphi^{(k)}(t)| \quad (p = 0, 1, \dots).$$

** In the case of several zeros of the function $\Delta_l(t)$ ($l = 1, 2$), P_l is replaced by the corresponding Hermite interpolation polynomial.

Proof. It is easy to see that $\mathfrak{A}^*\varphi = \mathfrak{M}_1\mathfrak{A}'\mathfrak{M}_2\varphi$ for any $\varphi \in C^\infty$, where $\mathfrak{M}_1, \mathfrak{M}_2$ are operators that map C^∞ onto itself one-to-one and continuously in both directions, and $\mathfrak{A}'$ is an operator of the form (3) with coefficients $A'(t), -B'(t)$ (the prime denotes the transposition operation). Thus, $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{A}'$, and together with them (see (2), Satz 13) also the operator $\mathfrak{A}_0^* = \mathfrak{A}^*$, considered on C^∞ , are Φ -operators. Consequently, the condition of Theorem 3 is satisfied.

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