

ON THE THEORY OF BOUNDARY-VALUE PROBLEMS FOR THE EQUATION

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Abstract

Full Text

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MATHEMATICS

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ON THE THEORY OF BOUNDARY-VALUE PROBLEMS FOR THE EQUATION

$y^m u_{xx} + u_{yy} = k(x, y)e^u$ IN THE LARGE

(Presented by Academician M. A. Lavrent'ev, 17 VI 1965)

Let D be a simply connected domain of the half-plane $y > 0$, bounded by the segment AB of the x -axis and by a curve σ , Hölder-continuous in the sense of a curve, which approaches the x -axis orthogonally at the points $A(-1, 0)$, $B(1, 0)$. Consider the equation

$$Eu(x, y) = y^m u_{xx} + u_{yy} = k(x, y)e^{u(x, y)}, \quad (1)$$

where m is a positive number, and $k(x, y)$ is a function nonnegative everywhere in the domain D , possessing continuous first derivatives in $\bar{D}$. Let $F(s)$ be a continuous function, prescribed on the boundary $\Gamma = \sigma + AB$ of the domain D , of the arc abscissa s , measured in the positive direction.

Theorem 1. *There exists a unique solution, twice continuously differentiable in the domain D , of equation (1), continuous in the closed domain $\bar{D}$ and satisfying the condition*

$$u(x, y)|_{\Gamma} = F(s). \quad (2)$$

In view of the fact that $k(x, y) > 0$ in the domain D , the uniqueness of the solution of problem (1)–(2) follows from the extremum principle ⁽¹⁾.

We shall carry out the proof of the existence of a solution according to the scheme proposed in ⁽²⁾.

The solution $u_0(x, y)$ of the equation

$$Eu(x, y) \equiv 0, \quad (3)$$

satisfying condition (2), exists and is unique. The existence of the Green function for problem (2)–(3) was proved in (3). In particular, in the case of the so-called normal domain, when the curve σ is given by the equation

$$x^2 + \frac{4}{(m+2)^2} y^{m+2} = 1, \quad y \geq 0,$$

the function $G(x, y; x_1, y_1)$ is constructed explicitly (3,4).

With the aid of the Green function, problem (1)–(2) is reduced to the equivalent integral equation

$$u(x, y) = - \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u(x_1, y_1)} dx_1 dy_1 + u_0(x, y). \quad (4)$$

Into equation (4) we introduce the parameter λ :

$$u(x, y) = -\lambda \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u(x_1, y_1)} dx_1 dy_1 + u_0(x, y). \quad (5)$$

A solution $u_\lambda(x, y)$ of equation (5) always exists for small values of the parameter $\lambda > 0$.

If we show the existence of a solution of equation (5) for $\lambda = 1$, then thereby the existence of a solution of problem (1)–(2) will be proved.

We note that, for any $\lambda > 0$, every solution of the integral equation (5) is bounded.

Indeed, by virtue of the inequality $G(x, y; x_1, y_1) > 0$, it follows from (5) that $u_\lambda(x, y) \leq u_0(x, y)$, i.e.

$$0 < e^{u_\lambda(x, y)} \leq e^{u_0(x, y)} \leq e^{\max u_0(x, y)} = e^{\max F},$$

and, consequently,

$$u_0(x, y) - \lambda \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{\max F} dx_1 dy_1 \leq u_\lambda(x, y) \leq u_0(x, y). \quad (6)$$

Suppose that the solution $u_\nu(x, y)$ of the integral equation (5) is known for $\lambda = \nu < 1$. Starting from this, we shall prove the existence of a solution $u_{\nu+\alpha}(x, y)$ of equation (5) for $\lambda = \nu + \alpha$, where α is some positive number.

It is not difficult to see that the function

$$\xi(x, y) = u_{\nu+\alpha}(x, y) - u_\nu(x, y)$$

satisfies the integral equation

$$\begin{aligned} \xi(x, y) + (\nu + \alpha) \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u_\nu(x_1, y_1)} \xi(x_1, y_1) dx_1 dy_1 \\ = -\alpha \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u_\nu(x_1, y_1)} dx_1 dy_1 \\ - (\nu + \alpha) \sum_{n=2}^{\infty} \frac{1}{n!} \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u_\nu(x_1, y_1)} \xi^n(x_1, y_1) dx_1 dy_1. \end{aligned} \quad (7)$$

By virtue of (6) we have

$$\max |\xi(x, y)| = \delta \leq \max \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{\max F} dx_1 dy_1 = d. \quad (8)$$

Since problem (2) with zero right-hand side for the equation

$$E\xi(x, y) = (\nu + \alpha) k(x, y) e^{u_\nu(x, y)} \xi(x, y)$$

is equivalent to the integral equation

$$\xi(x, y) + (\nu + \alpha) \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u_\nu(x_1, y_1)} \xi(x_1, y_1) dx_1 dy_1 = 0, \quad (9)$$

the latter cannot have nontrivial solutions. Hence, in turn, it follows that equation (7) is equivalent to the integral equation

$$\begin{aligned} \xi(x, y) = -\alpha \int_D R(x, y; x_1, y_1; \nu + \alpha) dx_1 dy_1 \\ - (\nu + \alpha) \sum_{n=2}^{\infty} \frac{1}{n!} \int_D R(x, y; x_1, y_1; \nu + \alpha) \xi^n(x_1, y_1) dx_1 dy_1, \end{aligned} \quad (10)$$

where $R(x, y; x_1, y_1; \nu + \alpha)$ is the resolvent of the kernel

$$(\nu + \alpha) G(x, y; x_1, y_1) k(x_1, y_1) e^{u_\nu(x_1, y_1)}.$$

Since the function

$$\int_D R(x, y; x_1, y_1; \nu + \alpha) dx_1 dy_1, \quad 0 \leq \nu + \alpha \leq 1,$$

is continuous in the closed domain $(x, y) \in \overline{D}$, the estimates

$$0 \leq \int_D R(x, y; x_1, y_1; \nu + \alpha) dx_1 dy_1 \leq B, \quad (11)$$

hold, where B is a constant.

From (8) and (11) follows the validity of the inequalities

$$(\nu + \alpha) \left| \sum_{n=2}^{\infty} \frac{1}{n!} \int_D R(x, y; x_1, y_1; \nu + \alpha) \zeta^n(x_1, y_1) dx_1 dy_1 \right| \leq B_1 \delta^2,$$

$$(\nu + \alpha) \left| \sum_{n=2}^{\infty} \frac{1}{n!} \int_D R(x, y; x_1, y_1; \nu + \alpha) \{ \zeta^n(x_1, y_1) - \bar{\xi}^n(x_1, y_1) \} dx_1 dy_1 \right| \leq B_2 \max |\zeta(x, y) - \bar{\xi}(x, y)|,$$

where $\bar{\xi}(x, y)$ is an arbitrary function of the class $C(\bar{D})$, $\max |\zeta(x, y)| < d$, and the positive constants B_1 and B_2 do not depend on ν and α .

According to these inequalities, from (10) we obtain

$$\delta \leq aB + B_1 \delta^2.$$

Let

$$\tau = 1 - \sqrt{1 - \alpha \cdot 4BB_1/2B_1}$$

be the smaller root of the quadratic equation

$$\tau = aB + B_1 \tau^2.$$

In the work ⁽²⁾ it is proved that if

$$B_2 \tau = q < 1, \tag{12}$$

then there exists a unique solution of the integral equation (10), satisfying the condition $|\zeta(x, y)| < \tau$.

Condition (12) will be satisfied for

$$a < (4B_2 - B_1)/4BB_2^2 = a^*. \tag{13}$$

This proves the existence of the solution $u_{\nu+\alpha}(x, y)$ for any positive $a < a^*$.

Repeating the above argument step by step, we reach the value $\lambda = 1$. Otherwise this process would break off for some value of the parameter $\lambda = \mu + \alpha$, i.e. the integral equation

$$\zeta(x, y) + (\mu + \alpha) \int_D G(x, y; x_1, y_1) k(x_1, y_1) e^{u_{\mu}(x_1, y_1)} \zeta(x_1, y_1) dx_1 dy_1 = 0$$

would have a nontrivial solution, which is impossible.

Thus the existence of a solution of problem (1)–(2) has been proved.

Theorem 2. *There exists a unique twice continuously differentiable in the domain D solution of equation (1), satisfying the conditions*

$$u(x, y)|_{\sigma} = \varphi(s), \quad (14)$$

$$\partial u(x, y)/\partial y|_{y=0} = \nu(x), \quad (15)$$

where $\varphi(s)$ is a prescribed continuous function on σ , and $\nu(x)$ is also a prescribed continuous function on the interval $0 < x < 1$, which may tend to infinity of order less than $2/(2+m)$ as $x \rightarrow 0$ or $x \rightarrow 1$.

The uniqueness of the solution of this problem again follows from the extremum principle.

Equation (3) has a unique solution satisfying conditions (14) and (15), which can be written with the help of the corresponding Green's function $G^*(x, y; x_1, y_1)$. For this problem the Green's function exists, and it is written explicitly in the case of a normal domain ^(5, 6).

The problem (1)–(14)–(15), with the aid of the Green's function, is reduced to a nonlinear integral equation, whose study is carried out analogously to the study of equation (5).

The boundary-value problems (2) and (14)–(15) for the equation

$$Eu(x, y) = f(x, y; u)$$

are studied analogously, under the condition that the function $f(x, y; u)$ can be expanded in a series in powers of $u(x, y)$, is positive everywhere in the domain D , is continuous in x, y , and the derivative $f'_u(x, y; u)$ is nonnegative in the domain D .

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