

On the Propagation of High-Frequency Signals in a Medium with Random Characteristics

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Abstract

Full Text

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PHYSICS

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ON THE PROPAGATION OF HIGH-FREQUENCY SIGNALS IN A MEDIUM WITH RANDOM CHARACTERISTICS

We consider an equation for the propagation of a signal Ψ of the form

$$A_{jk} \frac{\partial^2 \Psi}{\partial x_j \partial x_k} + B_k \frac{\partial \Psi}{\partial x_k} + C\Psi = 0, \quad (1)$$

where the coefficients A_{jk}, B_k, C are random functions of the variables x_j ($j = 1, 2, 3, 4$). It is assumed that, in the range of possible values of A_{jk} , equation (1) remains hyperbolic. Next put

$$A_{jk} = \bar{A}_{jk} + a_{jk}, \quad B_k = \bar{B}_k + b_k, \quad C = \bar{C} + c, \quad (2)$$

where the bar denotes averaging over possible values of the random quantities A_{jk}, B_k, C . This averaging has the meaning of functional integration over possible values of the random quantity $a(x)$:

$$\bar{\Phi} = \int \Phi\{a(x)\} dw\{a(x)\}, \quad (3)$$

where Φ is a functional of $a(x)$; $dw\{a(x)\}$ is the probability that $a = a(x)$. We shall assume that the random quantity $a(x)$ can be represented in the form of a series

$$a(x) = \sum_n a_n \varphi_n(x, \alpha_n), \quad (4)$$

where $\varphi_n(x, \alpha_n)$ is some system of orthonormal functions, α_n are random phases, and a_n are random amplitudes. In view of (4), $dw\{a(x)\}$ may be regarded as the

probability of one or another set of values of the quantities a_n, α_n ; in particular, if a_n, α_n are independent, then

$$dw\{a(x)\} = \prod_n dw(a_n) d\Omega(\alpha_n). \quad (5)$$

We shall seek the solution Ψ in the form

$$\Psi = Ae^{iS}, \quad (6)$$

where the frequency ω considerably exceeds the frequencies characteristic of the spectrum of the random quantities A_{jk}, B_k, C . In this case the amplitude A and the phase function S may be regarded as slowly varying functions of the variables x_j (the geometrical-optics approximation). Setting $S = \bar{S} + \sigma$ and substituting (6) into (1), as $\omega \rightarrow \infty$ we obtain:

$$\bar{A}_{jk} \frac{\partial \bar{S}}{\partial x_j} \frac{\partial \bar{S}}{\partial x_k} = 0, \quad (7)$$

$$\bar{A}_{jk} \frac{\partial \bar{S}}{\partial x_j} \frac{\partial \sigma}{\partial x_k} + \bar{A}_{jk} \frac{\partial \sigma}{\partial x_j} \frac{\partial \bar{S}}{\partial x_k} + a_{jk} \frac{\partial \bar{S}}{\partial x_j} \frac{\partial \bar{S}}{\partial x_k} = 0. \quad (8)$$

From the last equation one finds the random phase σ , which will be a linear functional of the random quantities $a_{jk}(x)$. Therefore $\sigma(x)$ has the form

$$\sigma(x) = \sum_n a_n \sigma_n(x, \alpha_n), \quad (9)$$

where $\sigma_n(x, a_n)$ corresponds to the solution of the system (7), (8), if in (4) all $a_m = 0$ are set, except for a_n .

The mean value of the signal Ψ will be

$$\bar{\Psi} = Ae^{i\omega \bar{S}} e^{i\omega \bar{\sigma}}. \quad (10)$$

If the distribution $dw(a_n)$ is normal,

$$dw(a_n) = \frac{1}{\sqrt{\pi}} \exp \left[-\frac{(a_n - \bar{a}_n)^2}{b_n^2} \right] \frac{da_n}{b_n}, \quad (11)$$

then, on the basis of (9), we obtain

$$\bar{\Psi} = Ae^{i\omega \bar{S}} \prod_n \int_0^{2\pi} d\Omega(\alpha_n) \exp \left[i\omega \bar{a}_n \sigma_n(x, \alpha_n) - \frac{b_n^2 \omega^2}{5} \sigma_n^2(x, \alpha_n) \right]. \quad (12)$$

The final result depends on the form of $\sigma(x, \alpha_n)$.

Let us consider several applications.

A. Scattering of sound in a turbulent flow. In the simplest case of a stationary, vortex-free flow, neglecting quantities of order u^2/c^2 (u is the flow velocity, c is the speed of sound), the equation for the velocity potential of the sound wave φ is ⁽¹⁾

$$\partial^2 \varphi / \partial t^2 + 2u \partial^2 \varphi / \partial x \partial t - c^2 \partial^2 \varphi / \partial x^2 = 0, \quad (13)$$

so that $x_1 = x$, $x_4 = t$ and $A_{44} = 1$, $A_{41} = a_{41} = 2u$, $A_{11} = -c^2$.

Equations (7) and (8) now have the form

$$\left(\frac{\partial \bar{S}}{\partial t} \right)^2 - c^2 \left(\frac{\partial \bar{S}}{\partial x} \right)^2 = 0, \quad (7')$$

$$\frac{\partial \bar{S}}{\partial t} \frac{\partial \sigma}{\partial t} - 2c^2 \frac{\partial \bar{S}}{\partial x} \frac{\partial \sigma}{\partial x} + 2u \frac{\partial \bar{S}}{\partial x} \frac{\partial \bar{S}}{\partial t} = 0. \quad (8')$$

From (7) for a plane wave we have $\bar{S} = t \pm x/c$, so that

$$\partial \sigma / \partial t \mp 2c \partial \sigma / \partial x + 2u/c = 0. \quad (14)$$

For a stationary flow $\partial u / \partial t = 0$, and one may obtain $\partial \sigma / \partial t = 0$. Then we obtain

$$\sigma(x) = \pm \int_0^x \frac{u(x') dx'}{c^2}. \quad (15)$$

Now putting

$$u(x) = \sum_n u_n \cos(q_{nx} + \alpha_n), \quad (16)$$

for the normal distribution law of u_n , with $\bar{u}_n = 0$, we obtain

$$\overline{e^{i\omega\sigma}} = \prod_n \int \exp \left[-\frac{\omega^2 b_n^2}{4q^2 c^4} F_n^2(x, \alpha_n) \right] \frac{d\alpha_n}{2\pi}, \quad (17)$$

where

$$F_n(x, \alpha_n) = \sin(q_{nx} + \alpha_n). \quad (18)$$

Since $F^2(x, \alpha_n) > 0$ and is bounded, this quantity in the exponent of (17) may be replaced by the effective mean $F_n^2(x, \theta_n \alpha_n)$, $0 < \theta_n < 1$. Then instead of (17) we obtain:

$$\overline{e^{i\omega\sigma}} \exp \left[-\frac{\omega^2}{2} \Phi(x) \right], \quad (19)$$

where

$$\Phi(x) = \sum_n \frac{b_n^2}{2q_n^2 c^4} F_n^2(x, \theta_n \alpha_n). \quad (20)$$

Thus, the mean strength of the sound signal (10) drops sharply with increasing frequency ω (according to a Gaussian curve).

B. Propagation of light in a medium with a turbulent metric.

The wave equation in this case is²

$$g^{\mu\nu} \partial^2 \varphi / \partial x_\mu \partial x_\nu - \Gamma^\mu \partial \varphi / \partial x_\mu = 0, \quad (21)$$

$$\Gamma^\mu = -\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_\nu} (\sqrt{-g} g^{\mu\nu}), \quad (22)$$

where g , as usual, is $\det(g_{\mu\nu})$.

We shall regard $g^{\mu\nu}$ as random functions of the variables $x_1, x_2, x_3, x_4 = t$. The physical reasons for such a possibility are considered in (3).

Let us turn to the case when

$$g^{\mu\nu} = g_0^{\mu\nu} + \varepsilon^{\mu\nu}, \quad \overline{\varepsilon^{\mu\nu}} = 0. \quad (23)$$

From (21), (22), (7), and (8) we obtain the equation for σ :

$$\varepsilon^{\mu\nu} \frac{\partial \bar{S}}{\partial x_\mu} \frac{\partial \bar{S}}{\partial x_\nu} + g_0^{\mu\nu} \left(\frac{\partial \bar{S}}{\partial x_\mu} \frac{\partial \sigma}{\partial x_\nu} + \frac{\partial \bar{S}}{\partial x_\nu} \frac{\partial \sigma}{\partial x_\mu} \right) = 0. \quad (24)$$

In particular, for a plane wave $\bar{S} = t - x$ (we take the speed of light c to be equal to 1), equation (24) takes the form

$$\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} = \frac{1}{2} \varepsilon(x, t), \quad \varepsilon = \varepsilon^{44} + \varepsilon^{11} - 2\varepsilon^{14}. \quad (25)$$

It has the solution

$$\sigma(x, t) = \frac{1}{4} \int_{t_0}^t \varepsilon(t' - \xi, t') dt' + \frac{1}{4} \int_{x_0}^t \varepsilon(x', \xi + x') dx', \quad (26)$$

where $\xi = t - x$. An analogous solution is obtained for the wave $\bar{S} = t + x$. If $\varepsilon(x, t)$ can be represented in the form

$$\varepsilon(x, t) = \sum_{n,m} \varepsilon_{nm} \sin(q_{nx} + \alpha_n) \sin(\omega_m t + \beta_m), \quad (27)$$

where $\varepsilon_{nm}, \alpha_n, \beta_m$ are random quantities, then, for a normal distribution law of ε_{nm} , $\overline{\varepsilon_{nm}} = 0$, $\overline{\varepsilon_{nm}^2} = \frac{1}{2} b_{nm}^2$, and a uniform distribution of α_n, β_m , we obtain

$$\sigma(x, t) = \sum_{n,m} \varepsilon_{nm} F_{nm}(x, t), \quad (28)$$

where the explicit form of $F_{nm}(x, t)$ is not difficult to obtain from (26) and (27). Applying the same reasoning as in point A, we find that the light signal has the form (19), with $\Phi(x) > 0$ now equal to

$$\Phi(x) = \frac{1}{2} \sum_{n,m} b_{nm}^2 F_{nm}^2(x, t, \theta_n \alpha_n, \theta_m \beta_m). \quad (29)$$

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Note: Figure translations are in progress. See original paper for figures.

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