

$\backslash(J\backslash)$ -NONEXPANDING OPERATORS IN A BANACH SPACE

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Abstract

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MATHEMATICS

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***J*-NONEXPANDING OPERATORS IN A BANACH SPACE**

(Presented by Academician L. S. Pontryagin, 5 XI 1965)

The present note adjoins our previous communication ⁽¹⁾ and uses the notation, definitions, and results from ⁽¹⁾.

1. In the Banach space $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$ ($\mathfrak{B}_\pm = P_\pm \mathfrak{B}$) one considers the *J*-metric

$$J(x) = \|P_+ x\|^2 - \|P_- x\|^2 \quad (x \in \mathfrak{B}). \quad (1)$$

In ⁽¹⁾ certain properties were established of linear operators V ($\mathfrak{D}_V \subset \mathfrak{B}$) possessing the property of *J*-nonnegativity: $J(Vx) \geq 0$ for $x \in \mathfrak{D}_V$ with $J(x) \geq 0$, or of *J*-nonpositivity: $J(Vx) \leq 0$ for $x \in \mathfrak{D}_V$ with $J(x) \leq 0$. We shall now consider in more detail one subclass of the class of *J*-nonpositive operators.

A linear operator V with $\mathfrak{D}_V \subset \mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$ will be called *J*⁻-nonexpanding if

$$J(Vx) \leq J(x) \quad (2)$$

for vectors $x \in \mathfrak{D}_V$ having the property $J(x) \leq 0$. If, however, inequality (2) is valid for all $x \in \mathfrak{D}_V$, then the operator V will be called simply *J*-nonexpanding.

In the case when $\mathfrak{B}$ is a Hilbert (or finite-dimensional unitary) space, and $\mathfrak{B}_+ \perp \mathfrak{B}_-$, i.e. $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$, *J*-nonexpanding operators have found wide application in the theory of stability of solutions of canonical systems of differential equations (see the works of M. G. Krein ⁽²⁻⁶⁾ and subsequent works developing these ideas—a detailed exposition of the question and a bibliography are given in ⁽⁷⁾). In turn, these works, in combination with the theory of characteristic matrix functions of M. S. Livshits ⁽⁸⁾, gave rise to a series of investigations in the theory of non-self-adjoint operators in Hilbert space, in which the notions of analytic *J*-nonexpanding matrix- and operator-functions began to play an essential role ⁽⁹⁻¹¹⁾; for details and bibliography see ⁽¹²⁾.

On the other hand, in the theory of operators in spaces with an indefinite metric, M. G. Krein ⁽¹³⁾ introduced and studied a more general class of J -nonexpanding operators*, though with the restriction that $\dim \mathfrak{B}_- = \infty$. Yu. P. Ginzburg ^(10,14) began to develop the general theory of J -nonexpanding operators in a Hilbert space $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$ (without restrictions on $\dim \mathfrak{B}_\pm$). In all these works the J -nonexpanding (and more general) operators were assumed a priori to be bounded and defined on the whole space.

* More precisely, in ⁽¹³⁾ the discussion concerned operators which in our terminology should be called J^+ -noncontracting ($\dim \mathfrak{B}_+ < \infty$).

In the present note we consider J - and J -nonexpanding operators defined on arbitrary lineals of a Banach space

$$\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-,$$

and are interested chiefly in questions of their closedness and boundedness. In doing so, some facts are found that had previously gone unnoticed even for J -isometric operators V ($J(Vx) = J(x)$, $x \in \mathfrak{D}_V$) in a Hilbert space.

2. We begin by setting out a number of simple properties of the operators of interest to us, for which purpose we introduce one more definition.

A lineal $\mathfrak{L}$ in $\mathfrak{B}$ will be called (cf. ⁽¹⁵⁾) **uniformly J -positive** (**uniformly J -negative**) if for all $x \in \mathfrak{L}$ we have

$$J(x) \geq c\|x\|^2$$

(respectively

$$J(x) \leq -c\|x\|^2), \quad c = \text{const} > 0.$$

The simplest examples of such lineals are $\mathfrak{B}_+$ (respectively $\mathfrak{B}_-$).

1°. A J -nonexpanding operator V carries every J -negative lineal $\mathfrak{L} \subset \mathfrak{D}_V$ into a J -negative lineal $V\mathfrak{L}$. Moreover, if the operator V is bounded, then every uniformly J -negative lineal $\mathfrak{L}$ is mapped by the operator V homeomorphically onto a uniformly J -negative lineal $V\mathfrak{L}$.

2°. The nullifying lineal

$$\mathfrak{Z} = \{x : x \in \mathfrak{D}_V, Vx = 0\}$$

of a J -nonexpanding operator V is J -nonpositive.

It is curious that, as examples show, this latter assertion (which follows directly from 1°) cannot, when $\mathfrak{D}_V \neq \mathfrak{B}$, be strengthened (i.e. one cannot assert that $\mathfrak{Z}$ is a J -positive lineal) either by imposing continuity of the operator V , or by passing from J -nonexpanding operators to J -isometric ones (even in a finite-dimensional unitary space). At the same time, for $\mathfrak{D}_V = \mathfrak{B}$ the following proposition of Yu. P. Ginzburg holds ^(14,15):

3°. The nullifying subspace $\mathfrak{Z}$ of a bounded J -nonexpanding operator defined on the whole Hilbert space

$$\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$$

is uniformly J -positive.

From 2° it follows that the nullifying lineal $\mathfrak{Z}$ of a J -isometric operator is J -neutral.

We shall call a J -isometric operator V **J -semiunitary** if $\mathfrak{D}_V = \mathfrak{B}$, and **J -unitary** if $\mathfrak{D}_V = \mathfrak{B}$ and $\mathfrak{R}_V \equiv V\mathfrak{D}_V = \mathfrak{B}$.

4°. If the domain $\mathfrak{D}_V$ of a J -isometric operator V is “decomposable,” i.e. is representable as a direct sum

$$\mathfrak{D}_V = \mathfrak{D}_+ \dot{+} \mathfrak{D}_-$$

of a J -positive ($\mathfrak{D}_+$) and a J -negative ($\mathfrak{D}_-$) lineal, then $\mathfrak{Z} = \{0\}$, i.e. the operator V is one-to-one on its (also decomposable) range $\mathfrak{R}_V$, and the inverse operator V^{-1} is again J -isometric. In particular, every J -semiunitary operator V is one-to-one, and if V is J -unitary, then V^{-1} is also J -unitary.

The last assertion immediately implies

5°. A J -unitary operator carries every maximal ⁽¹⁾ J -nonnegative (J -nonpositive) subspace into a maximal subspace of the same “sign.”

3. Inequality (2), proceeding from (1), can be rewritten in a form more convenient in many cases:

$$\|P_+ Vx\|^2 + \|P_- x\|^2 \leq \|P_- Vx\|^2 + \|P_+ x\|^2. \quad (3)$$

Using this inequality, it is not difficult to obtain the following strengthening of the general theorem 1 from ⁽¹⁾ for the special case of J -nonexpanding operators:

Theorem 1. For the boundedness of a J -nonexpanding operator V , defined on an arbitrary lineal $\mathfrak{D}_V (\subset \mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-)$, it is necessary and sufficient that the operator $P_- V$ be bounded.

In turn, theorem 2 from ⁽¹⁾ gives the following result for J -nonexpanding operators:

Theorem 2. If a J -nonexpanding operator V is defined on the whole Banach space $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$ and $(1 \leq) \varkappa = \dim \mathfrak{B}_- < \infty$, then the operator V is bounded.

As the examples show (see ⁽¹⁶⁾), the condition $\varkappa < \infty$ is essential in Theorem 2 even in the case of a Hilbert space $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$. As for the condition $\mathfrak{D}_V = \mathfrak{B}$, for a J -nonexpanding operator V it can be weakened by requiring only that $\mathfrak{D}_V$ contain a $\varkappa$ -dimensional J -negative subspace. This is not difficult to verify by comparing Theorem 1 of the present note with Theorem 2 of ⁽¹⁾.

4. Our further results rest on the following two lemmas, of which the second has a purely algebraic character, while the first contains, in addition to algebraic assertions, other assertions obtained with the aid of inequality (3).

Lemma 1. If V is a J -nonexpanding operator, then the operators $P_+ \dot{+} P_- V$ and $P_+ \dot{+} V P_-$ are one-to-one invertible. Moreover, if V is J -noncontracting, then the operators $(P_+ \dot{+} P_- V)^{-1}$ and $(P_+ \dot{+} V P_-)^{-1}$ are bounded.

Lemma 2. Let $\mathfrak{L} = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$ be an arbitrary (not necessarily normed) linear space, $\mathfrak{L}_\pm$ two of its subspaces, and $P_\pm$ the projectors from $\mathfrak{L}$ onto $\mathfrak{L}_\pm$, respectively, defined by the decomposition $\mathfrak{L} = \mathfrak{L}_+ \dot{+} \mathfrak{L}_-$. Let T be a linear operator given everywhere in $\mathfrak{L}$. The following three assertions are equivalent:

- 1) the operator $P_+ \dot{+} T P_-$ maps $\mathfrak{L}$ one-to-one onto all of $\mathfrak{L}$;
- 2) the operator $P_+ \dot{+} P_- T$ maps $\mathfrak{L}$ one-to-one onto all of $\mathfrak{L}$;
- 3) the operator $P_- T P_-$ maps $\mathfrak{L}_-$ one-to-one onto all of $\mathfrak{L}_-$.

Of course, the analogue of Lemma 2 is also valid in which the operators P_+ and P_- and the subspaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ are interchanged. In addition, in assertions 1) and 2) the sums of operators may be replaced by the differences of the same operators.

Theorem 3. In order that a J -nonexpanding operator V with $\mathfrak{D}_V \subset \mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$ be closed, it is necessary and sufficient that the operator $P_- V$ be closed.

From this theorem and Lemma 1 one obtains

Corollary 1. For a J -nonexpanding operator V to be closed, it is necessary that both lineals $(P_+ \dot{+} P_- V)\mathfrak{D}_V$ be closed, and sufficient that at least one of them be closed.

In combination with Theorem 1 this implies

Corollary 2. In the case where at least one of the two lineals $\mathfrak{D}_V$ and $(P_+ - P_- V)\mathfrak{D}_V$ is closed, for the boundedness of a J -nonexpanding operator V it is necessary and sufficient that the second lineal also be closed. In particular, every J -nonexpanding operator V defined everywhere in $\mathfrak{B}$ and such that $(P_+ - P_- V)\mathfrak{B} = \mathfrak{B}$ is bounded.

The last assertion of Corollary 2 is immediately connected with Lemma 2, from which it is seen that the condition $(P_+ - P_- V)\mathfrak{B} = \mathfrak{B}$ may be replaced by the equivalent condition (taking Lemma 1 into account) $P_- V\mathfrak{B}_- = \mathfrak{B}_-$, where the mapping $P_- V : \mathfrak{B}_- \rightarrow \mathfrak{B}_-$ is one-to-one. This fact plays a role in the theory of the so-called bi-nonexpanding J -nonexpanding operators in Hilbert space ^(10,14).

Using the terminology from ⁽¹⁾, the last result may be formulated also as follows:

6°. If V is a J -nonexpanding operator and $\mathfrak{D}_V = \mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, then for the boundedness of the operator V it is necessary and sufficient that the space $V\mathfrak{B}_-$ belong to the class $\mathfrak{T}_-$. This condition is always fulfilled, for example, for J -unitary operators (see 5°) in Hilbert space ⁽¹⁾, whence there follows a new proof of the known fact that all such operators are bounded ^(16,17).

In conclusion, we state a theorem, one special case of which was already reported by us in the form of a footnote in paper (16):

Theorem 4. In a Banach space $\mathfrak{B} = \mathfrak{B}_+ + \mathfrak{B}_-$, where $\dim \mathfrak{B}_- = \chi < \infty$, every J -nonexpanding operator V ($\mathfrak{D}_V \subset \mathfrak{B}$) is bounded if and only if it admits a closure.

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