

ON A CLASS OF FUNCTIONALS OF SAMPLE FUNCTIONS OF A STATIONARY GAUSSIAN PROCESS

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Abstract

Full Text

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MATHEMATICS

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ON A CLASS OF FUNCTIONALS OF SAMPLE FUNCTIONS OF A STATIONARY GAUSSIAN PROCESS

(Presented by Academician Yu. V. Linnik on 22 X 1965)

1. Let $\xi(t)$, $t \in [0, T]$, be a stationary Gaussian process with zero mathematical expectation and correlation function $R(\tau)$. Without loss of generality, we assume that $R(0) = 1$.

Everywhere below, by $f(x)$ (with an index or without an index) we denote Borel functions such that the integral

$$J = \int_0^T f(\xi(t)) dt \quad (1)$$

exists in the mean-square sense and, for all $m \geq 0$,

$$x^m e^{-x^2/2} f(x) \in L(-\infty, \infty).$$

In this note we study the structure of the closure (in the sense of convergence in mean square) of the class of functionals of the form (1) of the sample functions of the process $\xi(t)$.

2. By $H_m(x)$, $m = 0, 1, \dots$, we shall denote the normalized Chebyshev-Hermite polynomials

$$H_m(x) = (-1)^m (m!)^{-1/2} e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}.$$

The collection $\{H_m(x)\}$ forms a complete orthonormal system of polynomials with weight $\varphi(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$. Everywhere below

$$A_m^{(n_1, n_2)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f_{n_1}(x) - f_{n_2}(x)] H_m(x) e^{-x^2/2} dx. \quad (2)$$

Consider the two-dimensional Gaussian density

$$\varphi_{ts}(x, y) = \frac{1}{2\pi\sqrt{1-R^2(t-s)}} \exp \left\{ -\frac{x^2 + y^2 - 2R(t-s)xy}{2[1-R^2(t-s)]} \right\}.$$

It is known ⁽¹⁾ that for $\varphi_{ts}(x, y)$ the expansion

$$\varphi_{ts}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \sum_{m=0}^{\infty} R^m(t-s) H_m(x) H_m(y) \quad (3)$$

holds.

We shall need the following lemmas.

Lemma 1. Let

$$J_n = \int_0^T f_n(\xi(t)) dt, \quad n = 1, 2, \dots$$

For the existence of the limit l.i.m. _{$n \rightarrow \infty$} J_n^* it is necessary and sufficient that

* l.i.m. —limit in the mean-square sense.

was satisfied

$$\lim_{n_1, n_2 \rightarrow \infty} \sum_{m=0}^{\infty} p_m A_m^{(n_1, n_2)^2} = 0,$$

where

$$p_m = \int_0^T \int_0^T R^m(t-s) dt ds.$$

Lemma 2. Let the correlation function $R(\tau)$ be continuous, real-valued, positive, and let $R(\tau) \neq R(0)$ for $\tau \neq 0$. In addition, suppose that, for sufficiently small $\tau > 0$,

$$1 - C_2 \tau^\alpha \leq R(\tau) \leq 1 - C_1 \tau^\alpha,$$

for some constants $\alpha > 0$, $C_1 > 0$, $C_2 > 0$. Then there exist constants $K_1 > 0$ and $K_2 > 0$ such that for all $m = 1, 2, \dots$ one has

$$K_2 m^{-1/\alpha} \leq \int_0^T \int_0^T R^m(t-s) dt ds \leq K_1 m^{-1/\alpha}.$$

3. Let the functions $f_n(\xi(t))$ be such that $\mathbf{M}f_n(\xi(t)) = 0$ for all n . Then $A_0^{(n_1, n_2)} = 0$ for all n_1 and n_2 .

Theorem 1. Let the correlation function $R(\tau)$ of the stationary Gaussian process $\xi(t)$, $t \in [0, T]$, satisfy the conditions of Lemma 2. The limit l. i. m. $_{n \rightarrow \infty} J_n$ exists if and only if the limit exists

$$\text{l. i. m.}_{n \rightarrow \infty} g_n^{(\alpha)}(\xi(t)) = G_\alpha(\xi(t)), \quad (4)$$

where

$$g_n^{(\alpha)}(x) = \sqrt{2\pi e^{x^2/2}} \int_{-\infty}^{\infty} K_\alpha(x, y) f_n(y) dy,$$

$$K_\alpha(x, y) = \int_0^T \int_0^T \frac{1}{2\pi \sqrt{1 - Q_\alpha^2(t-s)}} \exp \left\{ -\frac{x^2 + y^2 - 2Q_\alpha(t-s)xy}{2[1 - Q_\alpha^2(t-s)]} \right\} dt ds,$$

$$Q_\alpha(\tau) = 1 - (|\tau|/T)^{2\alpha}.$$

Proof. From Lemmas 1 and 2 it follows that the limit l. i. m. $_{n \rightarrow \infty} J_n$ exists if and only if

$$\lim_{n_1, n_2 \rightarrow \infty} \sum_{m=1}^{\infty} \left(\frac{1}{m} \right)^{1/\alpha} A_m^{(n_1, n_2)^2} = 0. \quad (5)$$

Let us show that conditions (4) and (5) are equivalent. Using expansion (3) of the Gaussian density, we obtain

$$g_n^{(\alpha)}(x) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} H_m(x) q_m^{(\alpha)} \int_{-\infty}^{\infty} f_n(y) H_m(y) e^{-y^2/2} dy,$$

where

$$q_m^{(\alpha)} = \int_0^T \int_0^T Q_\alpha^m(t-s) dt ds.$$

It is easy to show that

$$\mathbf{M}(g_{n_1}^{(\alpha)}(\xi(t)) - g_{n_2}^{(\alpha)}(\xi(t)))^2 = \sum_{m=1}^{\infty} q_m^{(\alpha)^2} A_m^{(n_1, n_2)^2}.$$

It follows from this that, for the existence of the limit (4), it is necessary and sufficient that

$$\lim_{n_1, n_2 \rightarrow \infty} \sum_{m=1}^{\infty} q_m^{(\alpha)^2} A_m^{(n_1, n_2)^2} = 0. \quad (6)$$

From Lemma 2 it follows that, for some constants $K_1 > 0$ and $K_2 > 0$,

$$K_1 \sum_{m=1}^{\infty} \left(\frac{1}{m}\right)^{1/\alpha} A_m^{(n_1, n_2)^2} \leq \sum_{m=1}^{\infty} q_m^{(\alpha)^2} A_m^{(n_1, n_2)^2} \leq K_2 \sum_{m=1}^{\infty} \left(\frac{1}{m}\right)^{1/\alpha} A_m^{(n_1, n_2)^2}. \quad (7)$$

From (5), (6), and (7) the assertion of the theorem follows.

4. Condition (4) has a very simple meaning in the case where $1/\alpha$ is an integer.

Let $g_n^{(*k)}(x)$ denote the k -th antiderivative of the function $f_n(x)$, which satisfies the conditions

$$\mathbf{M}g_n^{(*i)}(\xi(t)) = 0 \quad (8)$$

for $i = 1, 2, \dots, k$. Thus, $g_n^{(*k)}(x)$ is determined uniquely.

Theorem 2. *Let the correlation function $R(\tau)$ of the stationary Gaussian process $\xi(t)$, $t \in [0, T]$, satisfy the conditions of Lemma 2 with $\alpha = 1/k$, where k is a positive integer. The limit l. i. m. $_{n \rightarrow \infty} J_n$ exists if and only if there exists the limit*

$$\text{l. i. m.}_{n \rightarrow \infty} g_n^{(*k)}(\xi(t)) = G_k^*(\xi(t)), \quad (9)$$

where $g_n^{(*k)}(x)$ is the above-defined k -th antiderivative of the function $f_n(x)$.

Proof. By repeated integration by parts it is not hard to show that

$$\mathbf{M}(g_{n_1}^{(*k)}(\xi(t)) - g_{n_2}^{(*k)}(\xi(t)))^2 = \sum_{m=k+1}^{\infty} \frac{A_{m-k}^{(n_1, n_2)^2}}{m(m-1) \dots (m-k+1)}.$$

Hence, for the existence of the limit (9), it is necessary and sufficient that the condition

$$\lim_{n_1, n_2 \rightarrow \infty} \sum_{m=1}^{\infty} \left(\frac{1}{m}\right)^k A_m^{(n_1, n_2)^2} = 0$$

be satisfied.

But from Lemmas 1 and 2 it follows that this same condition is the necessary and sufficient condition for the existence of the limit l. i. m. $_{n \rightarrow \infty} J_n$. The theorem is proved.

In conclusion, the author expresses his deep gratitude to A. V. Skorokhod for posing the problem and for valuable advice.

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References

1. G. V. L. Charlier, *Application de la théorie des probabilités à l'astronomie*, Paris, 1931.

Note: Figure translations are in progress. See original paper for figures.

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