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1966

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Abstract

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UDC 517.5

MATHEMATICS

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ON FRACTIONAL POWERS OF WEAKLY POSITIVE OPERATORS

(Presented by Academician M. A. Lavrent'ev, June 9, 1965)

1. In ⁽¹⁾ a new method was proposed for investigating fractional powers of a certain class of operators acting in the spaces L_p . It was established in which spaces L_{p_α} the operators $A^{-\alpha}$ act from L_{p_0} , if it is known in which L_{p_1} the operator A^{-1} acts from L_{p_0} . The proof is based on the moment inequality for fractional powers and the interpolation theorem from ⁽²⁾. In the present paper another method is proposed, which makes it possible to study the operators $BA^{-\alpha}$. The operators $BA^{-\alpha}$ were studied earlier in ⁽³⁾. However, restrictions were imposed there on the operator A , the validity of which in important cases of elliptic operators in domains has not yet been established.
2. An operator A , acting in a Banach space E , is called weakly positive (abbrev. w.p. E operator) if $D(A)$ is dense in E , the operator $A + tI$ has a bounded inverse for all $t \geq 0$, and $\|(A + tI)^{-1}\|_E \leq M(t+1)^{-1}$. Arbitrary powers of such operators are defined. In particular,

$$A^{-a} = \frac{\sin \pi(1-a)}{\pi(1-a)} \int_0^\infty t^{1-a} (A + tI)^{-2} dt \quad (0 < a < 2). \quad (1)$$

The moment inequality is valid

$$\|A^\alpha v\|_E \leq \frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)} M_1^{1-\alpha} M_2^\alpha \|Av\|_E^\alpha \|v\|_E^{1-\alpha} \quad (0 \leq \alpha \leq 1, v \in D(A)), \quad (2)$$

where

$$M_1 = \sup_{t>0} \|A(A + tI)^{-1}\|_E, \quad M_2 = \sup_{t>0} \|t(A + tI)^{-1}\|_E.$$

Fractional powers of w.p. E operators were first introduced and studied in ⁽⁴⁾ (see also ⁽⁵⁾).

Theorem 1. Let A be a w.p. E operator and let $F(v)$ be a seminorm continuous on $D(A)$. Suppose that for some $\alpha \in (0, 1)$ the inequality

$$F(v) \leq c \|Av\|_E^\alpha \|v\|_E^{1-\alpha} \quad (v \in D(A)) \quad (3)$$

holds. Then for every $\beta \in (\alpha, 1]$ the seminorm $F(v)$ can be extended to a seminorm $\bar{F}(v)$ continuous on $D(A^\beta)$, and for every $\gamma \in [0, \alpha)$ the inequality

$$\begin{aligned} \bar{F}(v) &\leq c \frac{\beta - \gamma}{(\beta - \alpha)(\alpha - \gamma)} \frac{\sin \pi(1 - \beta)}{\pi(1 - \beta)} \left[\frac{\sin \pi(\beta - \gamma)}{\pi(\beta - \gamma)(1 - \beta + \gamma)} \right]^{(\beta - \alpha)/(\beta - \gamma)} \times \\ &\times M_1^{\alpha + (\beta - \alpha)/(\beta - \gamma)} M_2^{2 - \alpha} \|A^\beta v\|_E^{(\alpha - \gamma)/(\beta - \gamma)} \|A^\gamma v\|_E^{(\beta - \alpha)/(\beta - \gamma)} \quad (v \in D(A^\beta)). \quad (4) \end{aligned}$$

Proof. Using successively (1), (3), and (2), we obtain that for every $N > 0$ on $D(A)$ the inequality

$$\begin{aligned} F(v) &\leq \frac{\sin \pi(1 - \beta)}{\pi(1 - \beta)} \int_0^\infty t^{1 - \beta} F([A + tI]^{-1} A^\beta [A + tI]^{-1} v) dt \leq \\ &\leq c \frac{\sin \pi(1 - \beta)}{\pi(1 - \beta)} M_1^\alpha M_2^{1 - \alpha} \left[\int_0^N t^{\alpha - \gamma - 1} \frac{\sin \pi(\beta - \gamma)}{\pi(\beta - \gamma)(1 - \beta + \gamma)} M_1 M_2 \|A^\gamma v\|_E dt + \right. \\ &\quad \left. + \int_N^\infty t^{\alpha - \beta - 1} M_2 \|A^\beta v\|_E dt \right]. \end{aligned}$$

Minimizing the square bracket with respect to N , we obtain (4).

Remark 1. If for some $0 \leq \gamma < \alpha < \beta \leq 1$ (4) is satisfied, then (3) follows easily from (2).

It follows from (4) that $\bar{F}(A^{-\beta} v)$ is continuous in E and

$$\bar{F}(A^{-\beta} v) \leq c(\alpha - \gamma, \beta - \gamma) \|v\|_E^{(\alpha - \gamma)/(\beta - \gamma)} \|A^{-(\beta - \gamma)} v\|_E^{(\beta - \alpha)/(\beta - \gamma)}. \quad (5)$$

The estimate of $\|A^{-\alpha} v\|_E$ follows from (2), if an estimate of $\|A^{-1} v\|_E$ is known (see (1)). In a more general situation one applies

Theorem 2. For arbitrary $0 \leq \gamma < \alpha < \beta \leq 1$ the inequality

$$\|A^{-\alpha} v\|_E \leq \frac{\sin \pi \alpha}{\pi} \frac{\beta - \gamma}{(\beta - \alpha)(\alpha - \gamma)} [\varphi_{\beta - \beta}(v)]^{(\alpha - \gamma)/(\beta - \gamma)} [\varphi_{1 - \gamma}(v)]^{(\beta - \alpha)/(\beta - \gamma)}, \quad (6)$$

$$\varphi_\delta(v) = \sup_{t>0} \|t^\delta(A + tI)^{-1}v\|_E.$$

Theorem 3. Let $\delta \in (0, 1)$, and let E_δ be a Banach space containing E . Let A be a w.p. E_δ operator, and suppose the inequality

$$\|v\|_E \leq c\|Av\|_{E_\delta}^\delta \|v\|_{E_\delta}^{1-\delta} \quad (v \in D(A) \subset E) \quad (7)$$

holds. Then the inequality

$$\varphi_{1-\delta}(v) \leq c[M_1(E_\delta)]^\delta [M_2(E_\delta)]^{1-\delta} \|v\|_{E_\delta} \quad (8)$$

holds.

3. Below we consider w.p. $L_p(\Omega)$ operators A , where Ω is a domain of n -dimensional space with boundary S . Inequality (5) makes it possible to estimate $\bar{F}(A^{-\beta}v_e)$ on characteristic functions of measurable sets $e \subset \Omega$, if an estimate of $\|A^{-(\beta-\gamma)}v_e\|_{L_p}$ is known. It is easy to see that, in any case,

$$\|A^{-(\beta-\gamma)}v_e\|_{L_p} \leq c(\text{mes } e)^{1/p}.$$

Such estimates, for the particular case of seminorms $\|BA^{-\beta}v\|_{L_q}$, where $L_q = L_q(G)$, and G is a domain of m -dimensional space and B is an operator from $L_p(\Omega)$ into $L_q(G)$, together with the interpolation theorem from (2), make it possible to prove the boundedness of the operators $BA^{-\beta}$.

Theorem 4. Let A be a w.p. $L_{p_i}(\Omega)$ operator, $i = 1, 2$. Suppose that for some $\varepsilon_0, \delta_{0,i} \in (0, 1)$

$$\|A^{-\varepsilon_0}v_e\|_{L_{p_i}} \leq c_i(\text{mes } e)^{\delta_{0,i}}. \quad (9)$$

Let B be a linear closed operator from $L_{p_i}(\Omega)$ into $L_{q_i}(G)$, $q_1 \neq q_2$, $D(B) \supset D(A)$, and

$$\|Bv\|_{L_{q_i}} \leq D_i \|Av\|_{L_{p_i}}^{\alpha_0} \|v\|_{L_{p_i}}^{1-\alpha_0} \quad (v \in D(A)), \quad (10)$$

where α_0 is some number in $(0, 1)$. Suppose β_0 satisfies the inequalities

$$\alpha_0 < \beta_0 < \min\{1, \alpha_0 + \varepsilon_0\}, \quad \frac{1}{r_i} \equiv \frac{1}{p_i} + \frac{\beta_0 - \alpha_0}{\varepsilon_0} \left(\delta_{0,i} - \frac{1}{p_i} \right) \geq \frac{1}{q_i},$$

and the numbers q_t and r_t , for arbitrary $t \in (0, 1)$, are defined by the equalities $1/q_t = t/q_1 + (1-t)/q_2$, $1/r_t = t/r_1 + (1-t)/r_2$. Then the operator $BA^{-\beta_0}$ admits a closure to a bounded operator from $L_{r_t}(\Omega)$ into $L_{q_t}(G)$.

Hence follows

Theorem 5. Let A satisfy the conditions of Theorem 4, and let R be a w.p. $L_{q_i}(G)$ operator. Suppose T and RTA^{-1} are bounded operators from $L_{p_i}(\Omega)$ into $L_{q_i}(G)$. Then the operator $R^{\alpha_0}TA^{-\beta_0}$ admits a closure to a bounded operator from $L_{r_t}(\Omega)$ into $L_{q_t}(G)$.

The proof is based on inequality (2), by means of which inequalities (10) are established for the operator $B = R^{\alpha_0}T$.

Remark 2. Let Q be a projection operator in $L_{p_i}(\Omega)$, and let A be a w.p. $E_i = QL_{p_i}(\Omega)$ operator. Then the operator $BA^{-\beta_0}Q$ from Theorem 4 and the operator $R^{\alpha_0}TA^{-\beta_0}Q$ from Theorem 5 admit a closure to a bounded operator from $L_{r_t}(\Omega)$ into $L_{q_t}(G)$.

4. Consider the boundary-value problem

$$Av + tv \equiv a(x; 1/iD)v(x) + (t_0 + t)v(x) = f(x) \quad (x \in \Omega), \quad (11)$$

$$B_{jv} \equiv B_j(x; 1/iD)v(x) = 0 \quad (x \in S, \quad j = 1, \dots, k).$$

Let $a_0(x; \xi)$ and $B_{j0}(x; \xi)$ be the collections of the highest-order terms of the polynomials $a(x; \xi)$ and $B_j(x; \xi)$, of degrees $2k$ and m_j , respectively. Let $a_0(x; \xi) + t^{2k} \neq 0$ for every real vector ξ and $t > 0$. Let, for any $x \in S$, vectors ξ and η (tangent and normal to S at the point x), the polynomial

$$p(z) = a_0(x; \xi + z\eta) + t^{2k}$$

have k roots z_i^+ with $\text{Im } z_i^+ > 0$. Let the polynomials $B_{j0}(x; \xi + z\eta)$ be linearly independent modulo the polynomial $(z - z_1^+) \dots (z - z_k^+)$. Finally, let the system of operators B_j be normal ⁽¹³⁾, and $m_j \leq 2k - 1$. Then ^(6,7) the following holds.

Theorem 6. For sufficiently large $t_0 > 0$ and for every $t \geq 0$, problem (11) is uniquely solvable in every $L_p(\Omega)$, and

$$t\|v\|_{L_p} + \|v\|_{W_p^{2k}} \leq c_p\|(A + tI)v\|_{L_p}. \quad (12)$$

From (12), first, it follows that A is a s.p. $L_p(\Omega)$ operator. Secondly, from (12) and the multiplicative inequalities for norms in the spaces of S. L. Sobolev–L. N. Slobodetskii (see ^(8,9)) there follows the inequality

$$\|v\|_{W_p^{2k\gamma}} \leq c\|Av\|_{L_{p\delta}}^\delta \|v\|_{L_{p\delta}}^{1-\delta} \quad (v \in D(A) \subset L_{p\delta}(\Omega)) \quad (13)$$

for any

$$0 \leq \gamma < 1, \quad \gamma < \delta < \min \left\{ 1, \frac{n}{2k} \left(1 - \frac{1}{p} \right) + \gamma \right\},$$

$$\frac{1}{p_\delta} = \frac{1}{p} + \frac{2k(\delta - \gamma)}{n}.$$

For $\gamma = 0$, from (13) we obtain an inequality of type (7), and for $\gamma \geq 0$, an inequality of type (10). When $2k\gamma = l + r$, l is an integer and $0 < r < 1$, one must consider the operator

$$Bv = |x - y|^{-n-r/p} [D_x^l v(x) - D_y^l v(y)],$$

acting from $D(A) \subset L_{p_\delta}(\Omega)$ into $L_p(\Omega \times \Omega)$. Thus one establishes

Theorem 7. $A^{-\delta}$ acts from $L_{p_\delta}(\Omega)$ into $W_p^{2k\gamma}(\Omega)$ and is bounded.

Only in the case when A is a self-adjoint operator in L_2 does the theory of scales ⁽¹⁰⁾ give a stronger result: the operator $A^{-\delta}$ acts from L_2 into $W_2^{2k\delta}$ and is bounded.

Remark 3. The conditions given above are satisfied by the operators A generated by the three classical boundary-value problems for elliptic operators of second order. The weak positivity of such operators was proved earlier in ⁽¹¹⁾.

Let A be an operator acting in $\dot{H}_p$, generated by the stationary Navier–Stokes equations (see ⁽¹²⁾). From Theorems 7 and 5 (see Remark 2) it follows that

Theorem 8. $A^{-\delta}$ acts from H_{p_δ} into $W_p^{2\gamma}$ and is bounded.

5. The operator A^* , acting in $L_{p'}(\Omega)$ ($1/p + 1/p' = 1$) and adjoint to the operator A acting in $L_p(\Omega)$, is generated by the adjoint differential expression a^* and the adjoint system of boundary conditions B_j^* , $j = 1, \dots, k$, which have the same properties as the differential operators a and B_j (see ⁽¹³⁾). Let $W_p^r(A)$ be the collection of functions $v(x) \in W_p^r(\Omega)$ satisfying the boundary conditions $B_{jv}(x) = 0$ for all $m_j \leq r - 1$ (r integer). Define $W_{p'}^r(A^*)$ analogously.

Theorem 9. Let $v \in W_p^{2k}(A)$, $u \in W_{p'}^r(A^*)$, $r = 0, 1, \dots, 2k$. Then

$$\left| \int_{\Omega} Av \bar{u} dx \right| \leq c(p, r) \|v\|_{W_p^{2k-r}} \|u\|_{W_{p'}^r}. \quad (14)$$

In the proof, constructions from ⁽¹³⁾ are used. Inequalities of the form (14) make it possible (see ^(5,14,15)) to investigate $D(A^\alpha)$.

Theorem 10. $W_p^{2k}(A) \supset D(A^\delta)$ in L_{p_δ} for integral $2k\gamma$, and $W_{p_\delta}^{2k\delta}(A) \subset D(A^\gamma)$ in L_p for integral $2k\delta$.

Here $\gamma, \delta, p, p_\delta$ are the same as in (13). Only in the case when A is a self-adjoint operator in L_2 , by means of the theory of scales¹⁰, is a stronger result established: $D(A^\delta) = W_2^{2k\delta}(A)$ for integer $2k\delta$.

Theorem 11. Let l and m be integers, $0 \leq l \leq 2k-1$, $1 \leq m \leq n$,

$$\frac{l}{2k} < \alpha < 1, \quad 0 < \delta < \left(\alpha - \frac{l}{2k}\right) \left(1 - \frac{l}{2k}\right)^{-1}, \quad \nu_\delta = \left(\alpha - \frac{l}{2k}\right) - \left(1 - \frac{l}{2k}\right) \delta,$$

$$p > \max \left\{ 1, \frac{n-m}{2k\nu_\delta} \right\}, \quad \frac{1}{p_\delta} = \frac{1}{p} \frac{m}{n} + \frac{2k\nu_\delta}{n},$$

and let G be a smooth manifold of dimension m lying in Ω . Then

$$\|A^{-\alpha}v\|_{W_p^l(G)} \leq c \|A^{-1}v\|_{W_{p_\delta}^l(\Omega)}^\delta \|v\|_{L_{p_\delta}(\Omega)}^{1-\delta}. \quad (15)$$

For the proof, in view of (1), one must estimate $\|(A+tI)^{-2}v\|_{W_p^l(G)}$ in two ways. First, from (13) there follows the estimate

$$\|(A+tI)^{-2}\|_{L_{p_\delta}(\Omega) \rightarrow W_p^l(G)}.$$

Second,

$$\|(A+tI)^{-2}v\|_{W_p^l(G)} \leq \|(A+tI)^{-1}\|_{L_{p_\delta}(\Omega) \rightarrow W_p^l(G)} \|(A+tI)^{-1}v\|_{L_{p_\delta}(\Omega)}.$$

The first factor is estimated by (13), and the second (see (14)) as follows:

$$\begin{aligned} \|(A+tI)^{-1}v\|_{L_{p_\delta}(\Omega)} &= \sup_u \left| \int_\Omega AA^{-1}v(A^*+tI)^{-1}u \, dx \right| \leq \\ &\leq c \|A^{-1}v\|_{W_{p_\delta}^l(\Omega)} \|(A^*+tI)^{-1}u\|_{W_{p'_\delta}^{2k-l}(\Omega)} \left(\frac{1}{p_\delta} + \frac{1}{p'_\delta} = 1, \quad \|u\|_{L_{p'_\delta}} = 1 \right). \end{aligned}$$

Next (13) is applied again.

Inequalities of type (15) make it possible to estimate $D^l A^{-\alpha}$, if an estimate is known for $D^l A^{-1}$. Attention is drawn to the latter circumstance in³.

6. To clarify the question of which spaces L_{p_α} from L_{p_0} the operators $A^{-\alpha}$ act in, it is convenient to use the following rule. The operators $A^{-\alpha}$ are similar in this sense to integral operators with kernel $|x-y|^{2k\alpha-n}$ (for $\alpha > n/2k$); the operators $D^l A^{-\alpha}$ ($l < 2k\alpha$) are similar to $A^{-\alpha+\beta}$, $\beta = l/2k$. For elliptic operators of second order this is not a formal rule, since it has been established¹⁶ that their fractional powers and derivatives of fractional powers are integral operators with kernels of the indicated type. Apparently, this fact also holds for operators of arbitrary order $2k$.

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Received
9 VI 1965

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